## 2.2. Homogenous Constant Coefficients Equations

**Section Objective(s):**

**Part 1:**
- Review: General and Fundamental Solutions.
- Guessing Fundamental Solutions for $2 \times 2$ Systems.
- Solutions for $2 \times 2$ Systems.

**Part 2:**
- Review: Solutions for $2 \times 2$ Systems.
- The Complex Roots Case.
- Real Solutions for Complex Roots.

**Remarks:**

- Recall:

**Theorem (General Solution).** If $y_1$, $y_2$, with $y_1 \neq c\, y_2$ for any $c \in \mathbb{R}$, are solutions of ________________ and ________________, where $L(y) = y'' + a_1\, y' + a_0\, y$, then ______________ solution $y$ of ____________________ can be written as

- Solutions $y_1$ and $y_2$ of $L(y) = 0$ with $y_1 \neq c\, y_2$ are called __________________________________________.
- If we know ________________________________________, then we know _________________________ of the ___________________________ equation.
- For _____________________ we _________ the __________________________ solutions.

### 2.2.1. **Guessing Fundamental Solutions for $2 \times 2$ Systems.**

Example 1. (Guessing Fundamental Solutions): Find all solutions to the equation

$$y'' + 5y' + 6y = 0.$$

Solution:

$\triangleleft$

**Definition 1.** The ***characteristic polynomial*** and ***characteristic equation*** of the differential equation

are, respectively,

**Theorem 1.** If $r_\pm$ are the roots of the characteristic polynomial of

if $c_+$, $c_-$ are arbitrary constants, then we have the following:

(a) If ________________, real or complex, then the general solution of Eq. (2.2.2) is

(b) If ______________________, real, then the general solution of Eq. (2.2.2) is

**Proof of Theorem 1:**

□

Example 2: Consider an object of mass $m = 1$ grams hanging from a spring with spring constant $k = 9$ grams per second square moving in a fluid with damping constant $d = 6$ grams per second. Find the position function of this object for arbitrary initial position and velocity.

Solution:

$\triangleleft$

Example 3: Find the solution $y$ of the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 5.$$

Solution:

$\triangleleft$

## Homogenous Constant Coefficients Equations

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**Part 1:**

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- Solutions for $2 \times 2$ Systems.

**Part 2:**

- Review: Solutions for $2 \times 2$ Systems.
- The Complex Roots Case.
- Review of Complex Numbers.

**Remarks:**

- Recall the $2 \times 2$ case:

**Theorem 1.** If $r_\pm$ are the roots of the characteristic polynomial of

(2.2.2)

and if $c_+$, $c_-$ are arbitrary constants, then we have the following:

(a) If ____________, real or complex, the general solution of Eq. (2.2.2) is

(b) If ____________________, real, the general solution of Eq. (2.2.2) is

- Equations with characteristic polynomial having ______________________________

  have ____________________________________.

- In some physical applications is important to have ____________________________.

- Solutions of equations with ________________________ describe

  ________________________________________.

### 2.2.2. **The Complex Roots Case.**

EXAMPLE 4: Consider an object of mass $m = 1$ grams hanging from a spring with spring constant $k = 13$ grams per second square moving in a fluid with damping constant $d = 4$ grams per second. Find the position function of this object for arbitrary initial position and velocity.

SOLUTION:

◁

### 2.2.3. **Review of Complex Numbers.**

Suppose that $a, b \in \mathbb{R}$. Then:

- Complex numbers have the form ________________, where _____________.
- The complex conjugate of $z$ is the number ________________.
- $\mathrm{Re}(z) = a$, $\mathrm{Im}(z) = b$ are the real and imaginary parts of $z$
- Hence: ____________________________, ____________________________.
- The **exponential of a complex number** is defined as

  ____________________________________________.

  In particular, the following is true: ________________________________.
- **Euler's formula:** ____________________________________________.
- Hence, a complex number of the form $e^{a+ib}$ can be written as
- From $e^{a+ib}$ and $e^{a-ib}$ we get the real numbers

**Theorem 2.** (Real Valued Fundamental Solutions) If the equation

has coefficients such that ______________________, then the roots of $p$ are complex,

and there are complex fundamental solutions of the differential equation,

while real valued fundamental solutions of the differential equation are

Furthermore, the general solution of the differential equation can be written either as

where $c_1$, $c_2$ are arbitrary constants, or as

where $A > 0$ is the _________________ and $\phi \in [-\pi, \pi)$ is the __________________.

**Proof of Theorem 2:**

□

Example 5. (Real Solutions - Mathematicians Notation): Describe the movement of the object in Example 4 above, which satisfies Newton's equation

$$y'' + 4y' + 13y = 0,$$

with initial position of 2 centimeters and initial velocity of 2 centimeters per second.

Solution:

◁

Example 6. (Real Solution - Physicists Notation): Write the solution of the Example 5 above in terms of the amplitude $A$ and phase shift $\phi$.

Solution:

$\triangleleft$