2.1. Scond Order Linear Equations: General Properties

Section Objective(s):

- Second Order Linear Equations.
- Conservation of Mechanical Energy.
- Properties of Homogeneous Equations.

Remarks:

- We now study __________________ order differential equations.
- The main example is ____.
- We have an ____________________________ about solutions these equations ____.
- We study ways to find __________________ of the solutions __________________________________ the equations.
- One way is with ____
- We end this section studying ____ equations.

2.1.1. Second Order Linear Equations.

Definition 1. A ***second order linear*** differential equation on y is

where a_1, a_0, b are given functions. The differential equation above:

(a) is ***homogeneous*** iff the source ______________ for all $t \in \mathbb{R}$;

(b) has ***constant coefficients*** iff ________________ are constants;

(c) has ***variable coefficients*** iff either ______________ is not constant.

Theorem 1. (IVP) If the a_1, a_0, b are continuous on (t_1, t_2) and $t_0 \in (t_1, t_2)$, then there is __________________________ solution of the initial value problem

Example 1. (Extendability of Solutions): Find the maximum domain where the solution of the initial value problem below is certain to exist.

$$(t-1)\, y'' - 3t\, y' + \frac{4(t-1)}{(t-3)}\, y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

Solution:

◁

2.1.2. **Conservation of Mechanical Energy.**

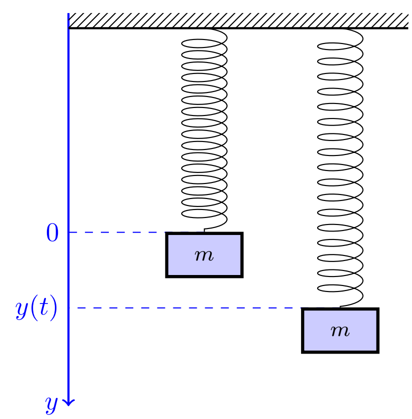
Remark: The main example of a second order linear equation is Newton's law of motion

The function y is the ______________, function $v = y'$ is the ______________,

and the function $a = y''$ is the ____________________ of a moving particle.

Example 2. (Mass-Spring System): Consider mass hanging at the bottom of a spring. **Hooke's Law** states the force is proportional to the stretching distance y

so the equation of motion of the mass is



Example 3. (Conservation of the Energy): Show that a mass-spring system moves keeping constant the quantity

Solution:

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EXAMPLE 4.: An object of mass $m = 1$ grams hanging at the bottom of a spring with a spring constant $k = 2$ grams per second square. Denote by y vertical coordinate, positive downwards, and $y = 0$ is the spring-mass resting position.

(1) Write the equation of motion for this object.

(2) Write the expression of the energy of this system.

(3) If the initial position of the object is $y(0) = 1$ and its initial velocity is $y(0) = 2$, find the maximum value of the object velocity, $v_{\max} > 0$ achieved during its motion.

SOLUTION:

$\triangleleft$

2.1.3. Properties of Homogeneous Equations.

Remark: We introduce the (operator) notation

Theorem 2.1.5. (Superposition Property) If y_1, y_2 are solutions of the homogeneous equations ________________ and ________________, where $L(y) = y'' + a_1\, y' + a_0\, y$, then for every constants c_1, c_2 holds

Remark: This result _______________________ for nonhomogeneous equations.

Proof:

□

Theorem (General Solution). If y_1, y_2, with $y_1 \neq c\, y_2$ for any $c \in \mathbb{R}$, are solutions of ________________ and ________________, where $L(y) = y'' + a_1\, y' + a_0\, y$, then _______________ solution y of ____________________ can be written as

Remark: Solutions y_1 and y_2 of $L(y) = 0$ with $y_1 \neq c\, y_2$ are called

____.

EXAMPLE 5. (SUPERPOSITION PROPERTY): If y_1 is solution of

$$y'' + a_1\, y' + a_0\, y = 0, \tag{1}$$

and $y - 2$ is solution of

$$y'' + a_1\, y' + a_0\, y = \cos(2t), \tag{2}$$

then determine whether the following statements are True or False.

(1) $y_1 + y_2$ solves the homogeneous equation (1)

(2) $y_1 + y_2$ solves the non-homogeneous equation (2)

(3) $2\, y_1$ solves the homogeneous equation (1)

(4) $2\, y_2$ solves the non-homogeneous equation (2)

SOLUTION:

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EXAMPLE 6. (FUNDAMENTAL SOLUTIONS): Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

SOLUTION:

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EXAMPLE 7.: Since $y_1 = 1$ is solution of

$$y'' + y' - 2y = -2.$$

find two more different solutions.

SOLUTION:

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