1.6. Approximate Solutions

Section Objective(s):

- The Existence of Solutions Theorem.
- The Picard Iteration.
- Linear vs Nonlinear Equations.

Remarks:

- If the equation ________________________, then ________________ solutions.
- The theorem is ____________ using __________________________________.
- The Picard iteration creates a ____.
- The solution of the equation is ____.
- We compare ____ linear and nonlinear equations.

1.6.1. **The Existence of Solutions Theorem.**

Theorem 1.3.1. (Picard-Lindelöf) Consider the initial value problem

If the function f and its partial derivative $\partial_y f$ are continuous on some rectangle on the ty-plane containing the point (t_0, y_0) in its interior,

then ____ of the initial value problem above on a smaller rectangle containing the condition (t_0, y_0).

Idea of the Proof:

$\square$

1.6.2. **The Picard Iteration.**

Example 1: Use three iterations of Picard's iteration procedure to find and approximate solution to

$$y' = 2\,y + 3 \qquad y(0) = 1.$$

Remark: We can compute the solution using the integrating factor method.

and the initial condition implies

Solution:

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1.6.3. Linear vs Nonlinear Equations.

Recall: The main theorem about solutions of linear equations.

Theorem 1. Given continuous functions a, b with domain (t_1, t_2), and constants $t_0 \in (t_1, t_2)$, $y_0 \in \mathbb{R}$, then the initial value problem

has the unique solution on the domain (t_1, t_2), given by

where ______________________________.

Solutions to linear equations satisfy:

(a) There is an ______________ formula for all solutions.

(b) ________________ initial condition y_0 there is a ______________ solution.

(c) ________________ IC y_0 the domain of $y(t)$ is ______________________.

Solutions to nonlinear equations satisfy:

(1) There is ______________________________ for the solution of

______________________________ differential equation.

(2) ________________ initial condition (t_0, y_0)

____.

(3) ________________ initial condition (t_0, y_0) the domain of the solution $y(t)$

______________________________.

Example 2. (Linear vs. Non-Linear ODEs): The solutions of the following equations are examples of the properties above. Identify which example corresponds to which property and explain your reasoning.

(1) $y'(t) = \dfrac{t^2}{\big(y^4(t) + 8\,y^3(t) + 9\,y^2(t) + 6\,y(t) + 7\big)}$.

(2) $y'(t) = y^{1/3}(t), \qquad y(0) = 0.$

(3) $y'(t) = y^2(t), \qquad y(0) = y_0.$

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EXAMPLE 3. (EXTENDABILITY OF SOLUTIONS TO LINEAR EQUATIONS): In the initial value problems below find the **maximum domain** where the solution is **certain** to exist.

(1) $t(t-5)\,y' = y, \qquad y(-1) = 4$

(2) $(t^2-4)\,y' - 5\ln(t)\,y = 3t, \qquad y(1) = 2$

SOLUTION:

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