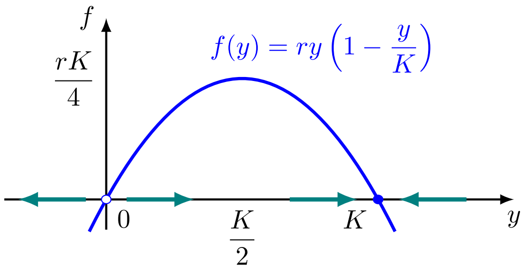
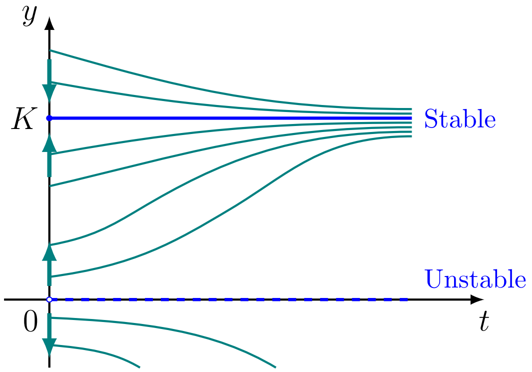
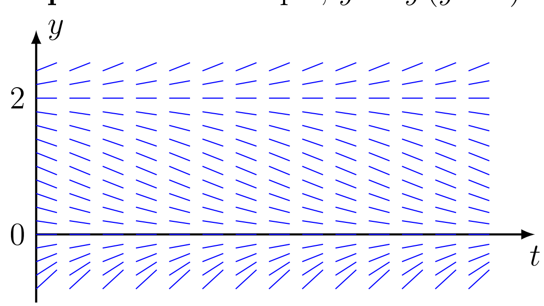
Autonomous Eqs. $y' = f(y)$.

Qualitative Solutions:
Example: $y' = ry\,(1 - y/K)$.





Slope fields: Example, $y' = y\,(y - 2)$.



Separable Eqs: $h(y)y = g(t)$.
Solution: $\int h(y)\,dy = c + \int g(t)\,dt$.

Euler Hom: $y' = F(y/t)$.
Transform to separable for: $v(t) = y(t)/t$.

Linear Eqs: $y' = a(t)\,y + b(t)$.

Integrating Factor: $\mu' = -a(t)\,\mu$

As a Total Derivative: $(\mu y)' = \mu\, b$.

General Solution for Const. Coeff.:

$$y(t) = c\,e^{at} - \frac{b}{a}, \quad c \in \mathbb{R}.$$

Newton Cooling Law: Constant medium temp. T_m. Object temp. $T(t)$.

$$\Delta T(t) = (T(t) - T_m), \quad \Delta T'(t) = -k\,\Delta T(t).$$

Mixing Problems:

$$V'(t) = (r_i - r_o)$$
$$Q'(t) = -\frac{r_o}{V(t)}\,Q(t) + r_i q_i.$$

Picard Iteration: $y_0(t) = y(0)$ and

$$y_{n+1}(t) = y(0) + \int_0^t f(s, y_n(s))\,ds.$$

SOLDE: $L(y) = b(t)$.
Operator: $L(y) = y'' + a_1\,y' + a_0\,y$.
Linearity: (c_1, c_2 constants)

$$L(c_1 y_1 + c_2 y_2) = c_1\,L(y_1) + c_2\,L(y_2).$$

Superposition:

$$L(y_1) = 0, \quad L(y_2) = 0 \quad \Rightarrow \quad L(c_1 y_1 + c_2 y_2) = 0.$$

General solutions of $L(y) = 0$,

$$y(t) = c_1\,y_1(t) + c_2\,y_2(t).$$

If $r_+ \neq r_-$ roots of $r^2 + a_1 r + a_0 = 0$,

$$y_1(t) = e^{r_+ t}, \quad y_2(t) = e^{r_- t}.$$

If $r_+ = r_- = r_0$ roots of $r^2 + a_1 r + a_0 = 0$,

$$y_1(t) = e^{r_0 t}, \quad y_2(t) = t\,e^{r_0 t}.$$

If $r_\pm = \alpha \pm \beta i$,

$$y_1(t) = e^{\alpha t}\cos(\beta t), \quad y_2(t) = e^{\alpha t}\sin(\beta t),$$

General solutions of $L(y) = f(t)$,

$$y(t) = c_1\,y_1(t) + c_2\,y_2(t) + y_{p_0}(t).$$

If $L(y_{p_0}) = 0$ then $y_{p_1} = t\,y_{p_0}$.
If $L(y_{p_1}) = 0$ then $y_{p_2} = t^2\,y_{p_0}$.

$$f(t) = 2\,e^{3t} \quad \Rightarrow \quad y_{p_0}(t) = k\,e^{3t}$$
$$f(t) = 2\,t^2 \quad \Rightarrow \quad y_{p_0}(t) = k_2\,t^2 + k_1\,t + k_0$$
$$f(t) = 2\sin(3t) \quad \Rightarrow \quad y_{p_0}(t) = k_1\cos(3t) + k_2\sin(3t)$$

Laplace Transform:

$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\, dt = F(s) = \mathcal{L}[f(t)](s)$$

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)} \qquad s > a$$
$$\mathcal{L}[\cos(at)] = \frac{s}{(s^2+a^2)} \qquad s > 0$$
$$\mathcal{L}[\sin(at)] = \frac{a}{(s^2+a^2)} \qquad s > 0$$
$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad s > 0.$$

Linearity: $\mathcal{L}[c_1 f + c_2 g] = c_1\,\mathcal{L}[f] + c_2\,\mathcal{L}[g]$

Derivative $\to$ Multiplication

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0)$$
$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0)$$

Multiplication $\to$ Derivative

$$\mathcal{L}[(-t)^n f(t)] = F^{(n)}(s)$$

Step function at c: $u(t-c)$.

Dirac's Delta at c: $\delta(t-c)$.

$$\mathcal{L}[u(t-c)] = \frac{e^{-cs}}{s}, \quad \mathcal{L}[\delta(t-c)] = e^{-cs}.$$

Translation Identities:

$$\mathcal{L}[u(t-c)\,f(t-c)] = e^{-cs}\,\mathcal{L}[f(t)]$$
$$\mathcal{L}[e^{ct}\,f(t)] = \mathcal{L}[f(t)](s-c)$$

Impulse Response: y_δ solution of

$$L(y) = \delta(t-c), \quad y(0) = 0, \quad y'(0) = 0.$$

where $L(y) = y'' + a_1 y' + a_0 y$.

Convolution: (Not Covered.)

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

Properties: $f * g = g * f$, $f * \delta = f$.

$$\mathcal{L}[f * g] = \mathcal{L}[f]\;\mathcal{L}[g]$$

Solution Decomposition: (Not Covered.) The solution y of

$$L(y) = f, \quad y(0) = y_0, \quad y'(0) = y_1$$

is $y(t) = y_h(t) + (y_\delta * f)(t)$, where y_h is

$$L(y_h) = 0, \quad y_h(0) = y_0, \quad y_h'(0) = y_1$$

Reduction to First Order: The SOLDE

$$m\,y'' + d\,y' + k\,y = f(t)$$

can be written as SFOLDE for the variables $x_1 = y$ and $x_2 = y'$, as follows

$$x_1' = x_2$$
$$x_2' = -\frac{k}{m}\,x_1 - \frac{d}{m}\,x_2 + f$$

Equilibrium Solutions of

$$x_1' = f_1(x_1, x_2), \quad x_2' = f_2(x_1, x_2),$$

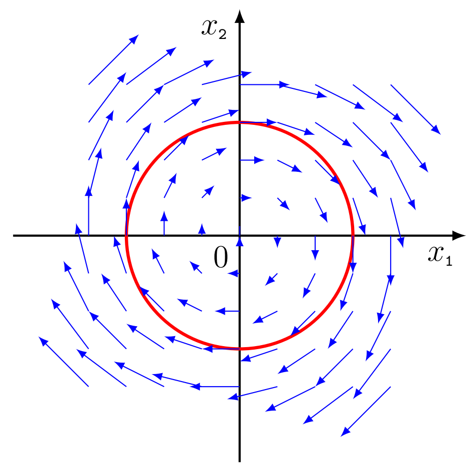
are constants x_1 x_2 so that

$$f_1(x_1, x_2) = 0, \quad f_2(x_1, x_2) = 0.$$

Vector Fields: The Mass-Spring $y'' + y = 0$ as first order system for $x_1 = y$, $x_2 = y'$ is

$$x_1' = x_2, \quad x_2' = -x_1.$$

Its vector field is $\vec{F}(x_1, x_2) = \langle x_2, -x_1 \rangle$. A solution curve is tangent to the vector field.



Matrix Operations: Multiplication,

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & -2 \end{bmatrix}$$

Given a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\det(A) = (ad - bc), \; A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eigenpairs: $A\boldsymbol{v} = \lambda \boldsymbol{v}$, with $\boldsymbol{v} \neq \boldsymbol{0}$

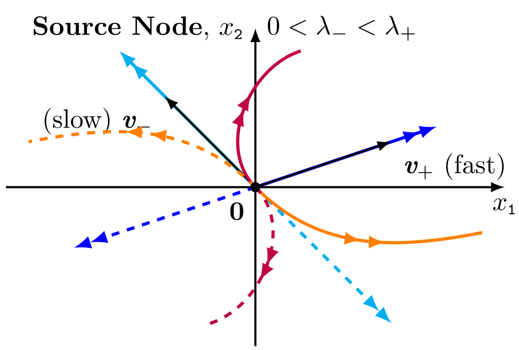
First, eigenvalues: $\det(A - \lambda I) = 0$.

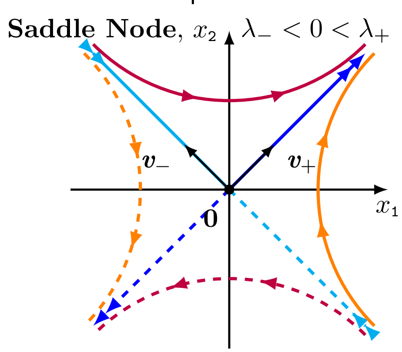
Second, eigenvectors: $(A - \lambda I)\boldsymbol{v} = \boldsymbol{0}$

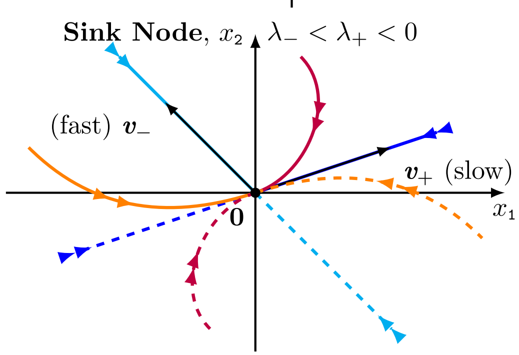
$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$, Real Eigenpairs

$$\lambda_+ \neq \lambda_-, \quad \boldsymbol{v}_+, \boldsymbol{v}_- \text{ l.i.}$$

$$\boldsymbol{x}_+(t) = \boldsymbol{v}_+ e^{\lambda_+ t}, \quad \boldsymbol{x}_-(t) = \boldsymbol{v}_- e^{\lambda_- t}.$$





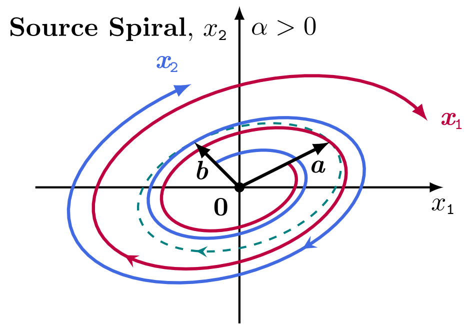


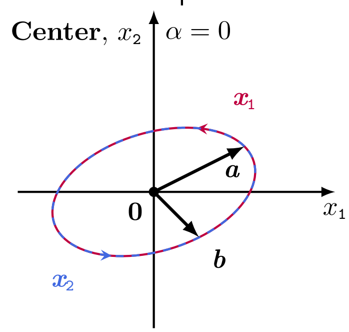
$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$, Complex Eigenpairs

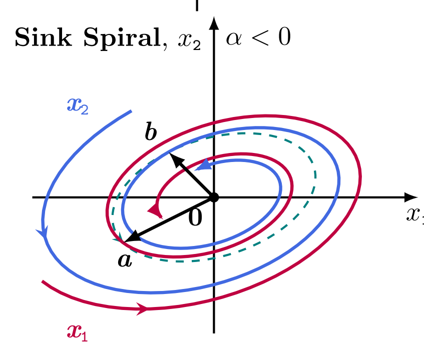
$\lambda_\pm = \alpha \pm \beta i$, $\boldsymbol{v}_\pm = \boldsymbol{a} \pm \boldsymbol{b} i$, rotation $\boldsymbol{a} \to -\boldsymbol{b}$.

$$\boldsymbol{x}_1(t) = e^{\alpha t}\big(\boldsymbol{a}\cos(\beta t) - \boldsymbol{b}\sin(\beta t)\big),$$

$$\boldsymbol{x}_2(t) = e^{\alpha t}\big(\boldsymbol{a}\sin(\beta t) + \boldsymbol{b}\cos(\beta t)\big),$$





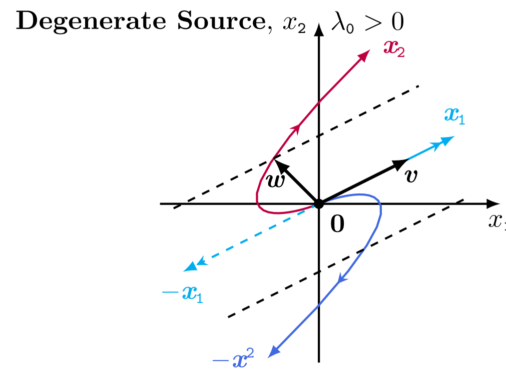


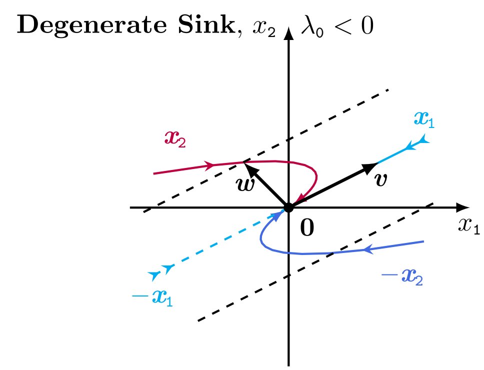
$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$, Repeated Eigenvalues

$$\lambda_+ = \lambda_- = \lambda_0, \quad \boldsymbol{v}_0, \quad \boldsymbol{w}$$

$$(A - \lambda_0 I)\boldsymbol{v}_0 = \boldsymbol{0}, \quad (A - \lambda_0 I)\boldsymbol{w} = \boldsymbol{v}_0$$

$$\boldsymbol{x}_1(t) = \boldsymbol{v}_0 e^{\lambda_0 t}, \quad \boldsymbol{x}_2(t) = (\boldsymbol{v}_0\, t + \boldsymbol{w})\, e^{\lambda_0 t}.$$





Note: The repeated eigenvalues graphs are not required for the exams. They are given here for completeness.

Nonlinear Systems $\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{x})$
Equilibrium Solutions: $\boldsymbol{f}(\boldsymbol{x}_c) = \boldsymbol{0}$.
Derivative Matrix:

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad Df = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix}.$$

If $D\boldsymbol{f}(\boldsymbol{x}_c)$ has eigenvalues $\lambda_\pm$ such that:

real, $0 < \lambda_- < \lambda_+$, then $\boldsymbol{x}_c$ source node;

real,$\lambda_- < \lambda_+ < 0$, then $\boldsymbol{x}_c$ sink node;

real, $\lambda_- < 0 < \lambda_+$, then $\boldsymbol{x}_c$ saddle node;

$\lambda_\pm = \alpha \pm \beta i$, $\alpha > 0$, then $\boldsymbol{x}_c$ source spiral.

$\lambda_\pm = \alpha \pm \beta i$, $\alpha < 0$, then $\boldsymbol{x}_c$ sink spiral.

$\lambda_\pm = \alpha \pm \beta i$, $\alpha = 0$, then $\boldsymbol{x}_c$ center.

Competing Species critical points:

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \boldsymbol{x}_1 = \begin{bmatrix} 0 \\ a \end{bmatrix}, \ \boldsymbol{x}_2 = \begin{bmatrix} b \\ 0 \end{bmatrix}, \ \boldsymbol{x}_3 = \begin{bmatrix} c \\ d \end{bmatrix}.$$

If $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are saddle nodes and $\boldsymbol{x}_3$ is a sink node, then **Coexistence**.

If $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ are sink nodes and $\boldsymbol{x}_3$ is a saddle node, then **Extinction**.

Predator-Prey critical points:

$$\boldsymbol{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \boldsymbol{x}_1 = \begin{bmatrix} a \\ b \end{bmatrix}.$$

If $\boldsymbol{x}_1$ is a center, then the populations **oscillate without damping**.

If $\boldsymbol{x}_1$ is a sink spiral, then the populations **oscillate and approach the** $\boldsymbol{x}_1$ as $t \to \infty$.

If $L(y)$ has $p(r)$ with **complex** roots, then the **BVP** for $L(y) = 0$ can have **infinitely** many solutions or **no solutions**.

Eigenfunction Problems $L(y) = \lambda y$ with BC are **BVP** with **infinitely** many solutions.

Fourier Series of $f(x)$ in $[-\ell, \ell]$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos\Big(\frac{n\pi x}{\ell}\Big) + b_n \sin\Big(\frac{n\pi x}{\ell}\Big) \Big)$$

$$a_0 = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x)\, dx$$

$$a_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \cos\Big(\frac{n\pi x}{\ell}\Big)\, dx$$

$$b_n = \frac{1}{\ell} \int_{-\ell}^{\ell} f(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx$$

Even Function iff $f_{\text{even}}(-x) = f_{\text{even}}(x)$.

Fourier Series of even functions don't have sines.

$$f_{\text{even}}(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{\ell}\Big)$$

$$a_0 = \frac{2}{\ell} \int_0^{\ell} f_{\text{even}}(x)\, dx$$

$$a_n = \frac{2}{\ell} \int_0^{\ell} f_{\text{even}}(x) \cos\Big(\frac{n\pi x}{\ell}\Big)\, dx$$

Odd Function iff $f_{\text{odd}}(-x) = -f_{\text{odd}}(x)$.

Fourier Series of odd functions don't have constant nor cosines.

$$f_{\text{odd}}(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{\ell}\Big)$$

$$b_n = \frac{2}{\ell} \int_0^{\ell} f_{\text{odd}}(x) \sin\Big(\frac{n\pi x}{\ell}\Big)\, dx$$

Heat Equation, Dirichlet BC

$$\left. \begin{aligned} u(t,0) &= 0 \\ u(t,\ell) &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} w(0) &= 0 \\ w(\ell) &= 0. \end{aligned} \right.$$

$$\lambda_n = \Big(\frac{n\pi}{\ell}\Big)^2, \ w_n(x) = \sin\Big(\frac{n\pi x}{\ell}\Big).$$

Heat Equation, Neumann BC

$$\left. \begin{aligned} \partial_x u(t,0) &= 0 \\ \partial_x u(t,\ell) &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} w'(0) &= 0 \\ w'(\ell) &= 0. \end{aligned} \right.$$

$\lambda_0 = 0$, $w_0(x) = 1$ and

$$\lambda_n = \Big(\frac{n\pi}{\ell}\Big)^2, \ w_n(x) = \cos\Big(\frac{n\pi x}{\ell}\Big).$$

Heat Equation, Mixed BC

$$\left. \begin{aligned} u(t,0) &= 0 \\ \partial_x u(t,\ell) &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} w(0) &= 0 \\ w'(\ell) &= 0. \end{aligned} \right.$$

$$\lambda_n = \Big(\frac{(2n-1)\pi}{2\ell}\Big)^2,$$

$$w_n(x) = \sin\Big(\frac{(2n-1)\pi x}{2\ell}\Big).$$

Heat Equation, Mixed BC

$$\left. \begin{aligned} \partial_x u(t,0) &= 0 \\ u(t,\ell) &= 0 \end{aligned} \right\} \Rightarrow \left\{ \begin{aligned} w'(0) &= 0 \\ w(\ell) &= 0. \end{aligned} \right.$$

$$\lambda_n = \Big(\frac{(2n-1)\pi}{2\ell}\Big)^2,$$

$$w_n(x) = \cos\Big(\frac{(2n-1)\pi x}{2\ell}\Big).$$