Review for Midterm Exam 2

Review for Chapter 4

(1) Consider a system of horses and mice, competing for the same resource, for example, grass. Assume the carrying capacity for horses is given by K_h and the carrying capacity for mice is K_m. In addition, the growth rate of horses is r_h and of mice is r_m. Finally, horses affect the mice population with an interaction coefficient α, and mice affect the horse population with interaction coefficient β.

(a) Write a system of differential equations modeling the horse and mice populations, $H(t)$ and $M(t)$.

(b) Based on the physical meaning of the coefficients, fill in the blanks with $>$ or $<$.

$$K_h \,\_\_\_\, K_m, \qquad r_h \,\_\_\_\, r_m, \qquad \alpha \,\_\_\_\, \beta.$$

(2) Consider a system of mosquitoes, swallows, and hawks. Assume the mosquitoes grow exponentially in the absence of swallows, and swallows feed on mosquitoes. Assume swallows would decrease exponentially without the mosquitoes and are hunted by hawks. Finally, assume hawks would decrease exponentially without swallows and reproduce at a rate proportional to their numbers and the food available.

Write the system of differential equations modeling the mosquito population $M(t)$, the swallow population $S(t)$, and hawk population $H(t)$.

(3) Consider the differential equation

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= -x(t). \end{aligned}$$

(a) Fid the vector field $\boldsymbol{F}(x, y)$ of the differential equation above.

(b) Draw the vector $\boldsymbol{F}(x, y)$ at the points $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$.

(c) In a separate picture sketch the direction field associated to the differential equation above.

(d) Use the direction field found above to sketch the solution curve of the system above corresponding to the initial data $x(0) = 0$, $y(0) = 1$.

Review for Chapter 5

(1) Sketch a graph of the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ -3 \end{bmatrix},$$

- **(a)** Does the linear system $\boldsymbol{v}_1\, x_1 + \boldsymbol{v}_2\, x_2 = \boldsymbol{b}$ have a solution? Determine if the solution is unique.
- **(b)** Write the above system in the form of two (scalar) equations and interpret the existence (or not) of a solution in terms of lines intersecting (or not).
- **(c)** Write the above system in matrix form. Determine if the matrix is invertible. If so, find the solution using the formula for matrix inverse.

(2) Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} h \\ 5 \end{bmatrix},$$

where h is a real number. Plot the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and use that plot to find all values of h such that the system below has a solution,

$$\boldsymbol{v}_1\, x_1 + \boldsymbol{v}_2\, x_2 = \boldsymbol{c}.$$

Also determine if the solution is unique.

(3) Determine which of the following sets is linearly independent.

$$S_A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}, \quad S_B = \left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -10 \end{bmatrix} \right\}, \quad S_C = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

- **(a)** If the set is linearly dependent, express one vector as a non-zero linear combination of the other vectors in the set.
- **(b)** If the set is linearly independent, show that the only linear combination of the above vectors which gives the zero vector is such that all scalars are zero.
- **(c)** For each of the sets, determine if the span of the vectors is the whole space, a plane, or a line.
- **(d)** For each of the sets, find a basis for their span.

(4) Consider the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 7 & 5 \\ 0 & 2 \end{bmatrix}.$$

Find the matrix X solution of the matrix equation $AXB + C = B$.

(5) Show that the matrix A below can be written as $A = PDP^{-1}$, with D diagonal.

$$A = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix}.$$

Also find e^{At} for any $t \in \mathbb{R}$.

Review for Chapter 6

(1) Find the solution to the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where the matrix A and the initial condition $\boldsymbol{x}_0$ are given below.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} -2 \\ -5 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$

(c) $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}.$

(2) Consider the following second order initial value problem

$$y'' + y' + 5y = -4\cos(5t), \qquad y(0) = -3, \qquad y'(0) = 2.$$

Write the problem above as a first order system of the form

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}(t), \qquad \boldsymbol{x} = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

(3) For each of the matrices below,

$$\text{(a)} \quad A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}, \qquad \text{(b)} \quad A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}, \qquad \text{(c)} \quad A = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}.$$

(i) Find the eigenvalues and eigenvectors of the matrix A.

(ii) Find a set of real-valued fundamental solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

(iii) Find the particular solution satisfying $\boldsymbol{x}(0) = \langle 1, 2 \rangle$.

(iv) Determine the type of equilibrium of the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. (Stable/unstable node, stable/unstable spiral, center, saddle.)

(v) Sketch a phase portrait of solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.