Review for Midterm Exam 2

Review for Chapter 4 - Hints and Solutions

Answer to **(1)**:

(a)

$$H'(t) = r_h\, H\Big(1 - \frac{H}{K_h}\Big) - \beta\, HM,$$
$$M'(t) = r_m\, M\Big(1 - \frac{M}{K_m}\Big) - \alpha\, HM.$$

(b) $K_h < K_m, \qquad r_h < r_m, \qquad \alpha > \beta.$

Answer to **(2)**:

$$M'(t) = \alpha M - \beta MS,$$
$$S'(t) = -\gamma S + \varepsilon MS - \kappa SH,$$
$$H'(t) = -\lambda H + \mu HS.$$

Answer to **(3)**: **(a)** $\boldsymbol{F}(x,y) = \langle -y, -x\rangle$.

(b)

$$\boldsymbol{F}(1,1) = \langle -1,-1\rangle, \quad \boldsymbol{F}(-1,1) = \langle -1,1\rangle, \quad \boldsymbol{F}(1,-1) = \langle 1,-1\rangle, \quad \boldsymbol{F}(-1,-1) = \langle 1,1\rangle.$$

(c) To sketch the direction field it is useful to also plot $\boldsymbol{F}$ at the following points:

$$\boldsymbol{F}(1,0) = \langle 0,-1\rangle, \quad \boldsymbol{F}(0,1) = \langle -1,0\rangle, \quad \boldsymbol{F}(-1,0) = \langle 0,1\rangle, \quad \boldsymbol{F}(0,-1) = \langle 1,0\rangle.$$

It also helps to write the system as $\boldsymbol{x}' = A\,\boldsymbol{x}$, with $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. This matrix A has eigenvalues $\lambda_{\pm} = \pm 1$.

(d) The solution curve belongs to the first and second quadrants, starts at $(0,1)$ for $t=0$, and notice that

$$\lim_{t\to\infty} \frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lim_{t\to-\infty} \frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Review for Chapter 5 - Hints and Solutions

Answer to **(1)**: **(a)** $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, so the solution exists and it is unique.

(b)

$$\begin{aligned} -x_1 + 3x_2 &= 7, \\ x_1 + 2x_2 &= -3. \end{aligned}$$

The two lines have different slopes, so they intersect at a single point, which means that the solution exists and it is unique.

(c)

$$\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Answer to **(2)**: The vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are parallel, so any linear combination of those two vectors will be a vector collinear with them. So, in order for the system

$$\boldsymbol{v}_1\, x_1 + \boldsymbol{v}_2\, x_2 = \boldsymbol{c}$$

to have a solution, we need $h = 5/2$. And in this case, the solution is **not** unique.

Answer to **(3)**: About Set S_A:

(a) S_A is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(b) Does not apply.

(c) The span of the three vectors is a plane.

(d) A basis for Span(S_A) is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

About Set S_B:

(a) The set S_B is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} = - \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$.

(b) Does not apply.

(c) The span of the three vectors is a line.

(d) A basis for Span(S_B) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$.

About Set S_C:

(a) Does not apply.

(b) The set S_C is linearly independent:

$$a\begin{bmatrix}-1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\3\\2\end{bmatrix}+c\begin{bmatrix}2\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow\quad a=b=c=0.$$

(c) The span of the three vectors is a the whole space.

(d) A basis for Span(S_C) is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\3\\2\end{bmatrix},\begin{bmatrix}2\\1\\-1\end{bmatrix}\right\}$.

Answer to **(4):** $X=\begin{bmatrix}-31 & 95\\ \frac{15}{2} & -23\end{bmatrix}$.

Answer to **(5):**

$$A=\begin{bmatrix}1&1\\2&1\end{bmatrix}\begin{bmatrix}2&0\\0&0\end{bmatrix}\begin{bmatrix}-1&1\\2&-1\end{bmatrix},\qquad e^{At}=\begin{bmatrix}(2-e^{2t}) & (-1+e^{2t})\\(2-2e^{2t}) & (-1+2e^{2t})\end{bmatrix}.$$

Review for Chapter 6 - Hints and Solutions

Answer to **(1)**: **(a)** $\boldsymbol{x}(t) = 7\,e^{3t}\begin{bmatrix}1\\-2\end{bmatrix} - 9\,e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$.

(b) $\boldsymbol{x}(t) = e^{2t}\begin{bmatrix}\cos(t)\\\sin(t)\end{bmatrix} + 4\,e^{2t}\begin{bmatrix}\sin(t)\\-\cos(t)\end{bmatrix}$.

(c) $\boldsymbol{x}(t) = 3\,e^{3t}\begin{bmatrix}1\\-1\end{bmatrix} - 8\,e^{3t}\Big(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}1\\0\end{bmatrix}\Big)$.

Answer to **(2)**: $A = \begin{bmatrix}0 & 1\\-5 & -1\end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix}0\\-4\cos(5t)\end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix}-3\\2\end{bmatrix}$.

Answer to **(3)**: **(a)** **(i)** $\lambda_{1,2} = 1 \pm 2i$, $\boldsymbol{v}_{1,2} = \begin{bmatrix}\mp i\\2\end{bmatrix}$.

(ii) $\boldsymbol{x}_1(t) = e^t\begin{bmatrix}\sin(2t)\\2\cos(2t)\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^t\begin{bmatrix}-\cos(2t)\\2\sin(2t)\end{bmatrix}$.

(iii) $\boldsymbol{x}(t) = e^t\begin{bmatrix}\sin(2t)+\cos(2t)\\2\cos(2t)-2\sin(2t)\end{bmatrix}$.

(iv) Unstable spiral.

(b) **(i)** $\lambda_1 = -2, \qquad \lambda_2 = 1, \qquad \boldsymbol{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$.

(ii) $\boldsymbol{x}_1(t) = e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^t\begin{bmatrix}1\\1\end{bmatrix}$.

(iii) $\boldsymbol{x}(t) = e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}$.

(iv) Saddle.

(c) **(i)** $\lambda_1 = -3, \qquad \lambda_2 = -1, \qquad \boldsymbol{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix}-1\\1\end{bmatrix}$.

(ii) $\boldsymbol{x}_1(t) = e^{-3t}\begin{bmatrix}-1\\2\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$.

(iii) $\boldsymbol{x}(t) = 3\,e^{-3t}\begin{bmatrix}-1\\2\end{bmatrix} - 4\,e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-3\,e^{-3t}+4\,e^{-t}\\6\,e^{-3t}-4\,e^{-t}\end{bmatrix}$.

(iv) Stable node.