Review for Final Exam

Review for Chapter 1

(1) Consider the population of worms in a composting pile. Assume the worm population increases by 20% each week and that a farmer takes 10 worms from the pile each week.

(a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the number of worms as a function of time.

(b) Assume that the initial worm population is 40 worms. Solve the ordinary differential equation in part (a) above with this given initial condition.

(c) Find the time t_1 when the farmer runs out of worms.

(2) A population of fish has a **growth rate proportional to the amount of fish present** at that time, with a proportionality factor of $\frac{1}{5}$ per unit time.

(a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the number of fish as a function of time.

(b) Now, assume that we have the same fish population, reproducing as above, but we are harvesting fish at a **constant rate of 100** fish per unit time. Write the differential equation in this case.

(c) Assume that the initial fish population is 600 fish. Solve the ordinary differential equation in part (b) above with this given initial condition.

(3) Use the Picard iteration to find the first 4 of a sequence $\{y_n\}$ of approximate solutions to the IVP

$$y'(t) = 8t^3 y(t), \quad y(0) = 4.$$

(4) Find the general solution to the following ODE

$$y' - 8y^2 \cos(t) - 7y^2 \sin(4t) = 0.$$

(5) A radioactive material has a **decay rate** proportional to the amount of radioactive material present at that time, with a proportionality factor of 2 per unit time.

(a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the amount of radioactive material (measured in micrograms) as a function of time.

(b) Now, assume that we have the same radioactive material decaying as above, but we are **adding** additional material (of the same type) **at a constant rate of 6** micrograms per unit time. Write the differential equation in this case.

(c) Solve the ordinary differential equation in part (b) above, assuming the initial amount of radioactive material is 70 micrograms.

(6) Find the solution to the following initial value problem

$$y' + 8y^3 \cos(7t) = 0, \qquad y(0) = 2.$$

(7) Find the solution to the following IVP

$$ty' = 2y - 3t^3\cos(4t), \qquad y(\pi/8) = 0.$$

(8) Consider the differential equation

$$\frac{dy}{dt} = y(y^2 - 4)(y^2 + 9).$$

(a) Find the **equilibrium solutions** of the ODE.

(b) Construct a phase diagram and determine the stability of the critical points.

(c) Make rough sketches of typical solution curves.

(9) Find the solution to the following IVP

$$y' = \tan(t)y - 5t, \quad t \in [0, \frac{\pi}{2}), \qquad y(0) = 3.$$

(10) Find an explicit expression for the solution y of the following initial value problem.

$$y' = \frac{3y^3 + t^3}{ty^2}, \quad y(1) = 2, \quad t \geq 1.$$

(11) A glass of cold soda is placed into a room held at 30 C.

(a) If k is a (positive cooling constant), find the differential equation satisfied by the temperature, $T(t)$ of the soda.

(b) Find the soda temperature as a function of time (and k), if the initial temperature of the soda was 2 C.

(c) If after 40 minutes the soda temperature was 10 C, find the cooling constant k.

Review for Chapter 2

(1) An object of mass 2 gr is hanging at the bottom of a spring with a spring constant 3 gr/sq.sec. Let $y(t)$ denote the vertical coordinate, positive downwards and $y = 0$ be the resting position. Find the mechanical energy of the system. If the initial position of the object is $y(0) = -3$ and its initial velocity is $y'(0) = 3$, find the maximum value of the position of the object, achieved during this motion.

(2) Find the general solution of

$$y'' - 8y' + 16y = 0.$$

(3) Solve the initial value problem

$$y'' - 5y' + 4y = 0, \qquad y(0) = -5, \quad y'(0) = 3.$$

(4) Solve the initial value problem

$$y'' - 8y' + 32y = 0, \qquad y(0) = -2, \quad y'(0) = -4.$$

(5) Solve the initial value problem

$$y'' - 8y' + 15y = 4e^t, \qquad y(0) = 5, \quad y'(0) = 1.$$

(6) Find the general solution of

$$y'' - 6y' + 8y = 3e^{2t}.$$

(7) Find the general solution of

$$y'' - 6y' + 9y = 4\,e^{3t}.$$

(8) Find the general solution of

$$y'' - 10y' + 24y = -3\sin(2t).$$

Review for Chapter 3

(1) Use the Laplace transform to solve the initial value problem

$$y'' + 6y' + 10y = 0, \quad y(0) = -5, \quad y'(0) = -4.$$

(2) Solve the initial value problem

$$y'' - 8y' + 16y = 5\delta(t-3), \quad y(0) = 0, \quad y'(0) = 0.$$

(3) Consider the following second order IVP with an arbitrary force term, $g(t)$

$$y'' - 4y' + 20y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Let $G(s) = \mathcal{L}[g]$ and $Y(s) = \mathcal{L}[y]$. Find $H(s)$, such that $Y(s) = H(s)G(s)$ and $h(t)$ such that $y(t) = h \star g(t)$.

(4) Find the Laplace transform of the function

$$f(t) = \begin{cases} 0, & t < 3 \\ t^2 - 6t + 7, & t \geq 3. \end{cases}$$

(5) Solve the initial value problem

$$y'' - 7y' + 12y = 5u(t-3)e^{-3t}, \quad y(0) = 0, \quad y'(0) = 0.$$

(6) Solve the initial value problem

$$y'' - 5y' + 4y = -5u(t-9), \quad y(0) = 0, \quad y'(0) = 0.$$

(7) Solve the initial value problem

$$y'' - 7y' + 6y = e^{5t}\delta(t-5), \quad y(0) = 0, \quad y'(0) = 0.$$

Review for Chapter 4

(1) Consider a system of horses and mice, competing for the same resource, for example, grass. Assume the carrying capacity for horses is given by K_h and the carrying capacity for mice is K_m. In addition, the growth rate of horses is r_h and of mice is r_m. Finally, horses affect the mice population with an interaction coefficient α, and mice affect the horse population with interaction coefficient β.

(a) Write a system of differential equations modeling the horse and mice populations, $H(t)$ and $M(t)$.

(b) Based on the physical meaning of the coefficients, fill in the blanks with $>$ or $<$.

$$K_h \,\_\_\_\, K_m, \qquad r_h \,\_\_\_\, r_m, \qquad \alpha \,\_\_\_\, \beta.$$

(2) Consider a system of mosquitoes, swallows, and hawks. Assume the mosquitoes grow exponentially in the absence of swallows, and swallows feed on mosquitoes. Assume swallows would decrease exponentially without the mosquitoes and are hunted by hawks. Finally, assume hawks would decrease exponentially without swallows and reproduce at a rate proportional to their numbers and the food available.

Write the system of differential equations modeling the mosquito population $M(t)$, the swallow population $S(t)$, and hawk population $H(t)$.

(3) Consider the differential equation

$$\begin{aligned} x'(t) &= -y(t) \\ y'(t) &= -x(t). \end{aligned}$$

(a) Fid the vector field $\boldsymbol{F}(x, y)$ of the differential equation above.

(b) Draw the vector $\boldsymbol{F}(x, y)$ at the points $(1, 1)$, $(-1, 1)$, $(1, -1)$, $(-1, -1)$.

(c) In a separate picture sketch the direction field associated to the differential equation above.

(d) Use the direction field found above to sketch the solution curve of the system above corresponding to the initial data $x(0) = 0$, $y(0) = 1$.

Review for Chapter 5

(1) Sketch a graph of the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 7 \\ -3 \end{bmatrix},$$

(a) Does the linear system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{b}$ have a solution? Determine if the solution is unique.

(b) Write the above system in the form of two (scalar) equations and interpret the existence (or not) of a solution in terms of lines intersecting (or not).

(c) Write the above system in matrix form. Determine if the matrix is invertible. If so, find the solution using the formula for matrix inverse.

(2) Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}, \qquad \boldsymbol{c} = \begin{bmatrix} h \\ 5 \end{bmatrix},$$

where h is a real number. Plot the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and use that plot to find all values of h such that the system below has a solution,

$$\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{c}.$$

Also determine if the solution is unique.

(3) Determine which of the following sets is linearly independent.

$$S_A = \left\{ \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}, \quad S_B = \left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ -10 \end{bmatrix} \right\}, \quad S_C = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

(a) If the set is linearly dependent, express one vector as a non-zero linear combination of the other vectors in the set.

(b) If the set is linearly independent, show that the only linear combination of the above vectors which gives the zero vector is such that all scalars are zero.

(c) For each of the sets, determine if the span of the vectors is the whole space, a plane, or a line.

(d) For each of the sets, find a basis for their span.

(4) Consider the matrices

$$A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}, \qquad B = \begin{bmatrix} -2 & 6 \\ -1 & 2 \end{bmatrix}, \qquad C = \begin{bmatrix} 7 & 5 \\ 0 & 2 \end{bmatrix}.$$

Find the matrix X solution of the matrix equation $AXB + C = B$.

(5) Show that the matrix A below can be written as $A = PDP^{-1}$, with D diagonal.

$$A = \begin{bmatrix} -2 & 2 \\ -4 & 4 \end{bmatrix}.$$

Also find e^{At} for any $t \in \mathbb{R}$.

Review for Chapter 6

(1) Find the solution to the initial value problem

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

where the matrix A and the initial condition $\boldsymbol{x}_0$ are given below.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} -2 \\ -5 \end{bmatrix}.$

(b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}.$

(c) $A = \begin{bmatrix} 4 & 1 \\ -1 & 2 \end{bmatrix}, \qquad \boldsymbol{x}_0 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}.$

(2) Consider the following second order initial value problem

$$y'' + y' + 5y = -4\cos(5t), \qquad y(0) = -3, \qquad y'(0) = 2.$$

Write the problem above as a first order system of the form

$$\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t) + \boldsymbol{b}(t), \qquad \boldsymbol{x} = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

(3) For each of the matrices below,

$$(\mathbf{a}) \quad A = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}, \qquad (\mathbf{b}) \quad A = \begin{bmatrix} 4 & -3 \\ 6 & -5 \end{bmatrix}, \qquad (\mathbf{c}) \quad A = \begin{bmatrix} 1 & 2 \\ -4 & -5 \end{bmatrix}.$$

(i) Find the eigenvalues and eigenvectors of the matrix A.

(ii) Find a set of real-valued fundamental solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

(iii) Find the particular solution satisfying $\boldsymbol{x}(0) = \langle 1, 2\rangle$.

(iv) Determine the type of equilibrium of the trivial solution $\boldsymbol{x} = \boldsymbol{0}$. (Stable/unstable node, stable/unstable spiral, center, saddle.)

(v) Sketch a phase portrait of solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

(4) Given the following nonlinear system of differential equations, find all equilibrium points, find the matrix of the linearization around each equilibrium, and determine the equilibrium's type and stability.

$$\begin{aligned} x' &= 4y - y^3 \\ y' &= -9x - y^2. \end{aligned}$$

Review for Chapter 7

(1) Find all solutions of the boundary value problem $y'' + 9\,y = 0$, with boundary conditions

(a) $y(0) = 2$, $y(\pi/2) = 1$;

(b) $y(0) = 2$, $y(\pi) = -2$;

(c) $y(0) = 2$, $y(\pi) = 2$.

(2) Find the eigenvalues λ_n and corresponding (nonzero) eigenfunctions y_n, which solve

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(4) = 0.$$

(3) Find the Fourier series, f_F, of the following function

$$f(x) = 2x + 5, \qquad x \in [-3, 3].$$

(4) Consider the function $f(x) = 2x + 1$ defined on $(0, 3]$ and extend it to an **even** function $f_e(x)$ defined on $[-3, 3]$. Find the Fourier series expansion, f_{eF}, of the function $f_e(x)$.

(5) Let u be the solution to the following initial boundary value problem for the Heat Equation

$$\partial_t u(t,x) = 3\,\partial_x^2 u(t,x), \quad t > 0, \quad x \in (0,3),$$

with an initial condition $u(0,x) = f(x)$ and with boundary conditions

$$\partial_x u(t,0) = 0, \qquad \partial_x u(t,3) = 0.$$

Find functions $v_n(t)$ and $w_n(x)$ such that $u(t,x) = \sum_{n=1}^{\infty} c_n v_n(t) w_n(x)$.

(6) Consider the conduction of heat in a 40 cm in long rod with conductivity constant $k = 1$, whose ends are maintained at 0°C for all $t > 0$. Find an expression for the temperature $u(t,x)$ if the initial temperature distribution in the rod is given by

$$u(0,x) = \begin{cases} 0, & 0 \le x < 10, \\ 50, & 10 \le x \le 30, \\ 0, & 30 < x \le 40. \end{cases}$$