

REVIEW FOR FINAL EXAM

Review for Chapter 1 - Hints and Solutions

Answer to (1): (a) $P'(t) = \frac{1}{5}P(t) - 10$,

(b) $P(t) = -10e^{t/5} + 50$,

(c) $t_1 = 5 \ln(5)$.

Answer to (2): (c): $P(t) = 500 + 100e^{t/5}$.

Answer to (3): Partial Answer: $y_3(t) = 4 + 8t^4 + 8t^8 + \frac{16}{3}t^{12}$.

Answer to (4): $y(t) = \left(-8 \sin(t) + \frac{7}{4} \cos(4t) + c\right)^{-1}$

Answer to (5): (c): $P(t) = 3 + 67e^{-2t}$.

Answer to (6): $y(t) = \left(\frac{16}{7} \sin(7t) + \frac{1}{4}\right)^{-1/2}$

Answer to (7): $y(t) = \frac{3}{4}t^2(1 - \sin(4t))$

Answer to (8): Part of (b): critical pts: -2 (unstable), 0 (stable) , 2 (unstable).

Answer to (9): $y(t) = \frac{8}{\cos(t)} - 5t \tan(t) - 5$

Answer to (10): $y(t) = t \left(\frac{17}{2}t^6 - \frac{1}{2}\right)^{1/3}$.

Answer to (11): (a) $T' = -k(T - 30)$;

(b) $T(t) = 30 - 28e^{-kt}$;

(c) $k = \frac{1}{40} \ln\left(\frac{7}{5}\right)$.

Review for Chapter 2 - Hints and Solutions

Answer to **(1)**: $E = (y')^2 + \frac{3}{2}y^2$; $y_{max} = \sqrt{15}$.

Answer to **(2)**: $y(t) = c_1 e^{4t} + c_2 t e^{4t}$.

Answer to **(3)**: $y(t) = \frac{8}{3}e^{4t} - \frac{23}{3}e^t$.

Answer to **(4)**: $y(t) = -2e^{4t} \cos(4t) + e^{4t} \sin(4t)$.

Answer to **(5)**: $y(t) = -\frac{13}{2}e^{5t} + 11e^{3t} + \frac{4}{8}e^t$.

Answer to **(6)**: $y(t) = c_1 e^{4t} + c_2 e^{2t} - \frac{3}{2}t e^{2t}$.

Answer to **(7)**: $y(t) = c_1 e^{3t} + c_2 t e^{3t} + 2t^2 e^{3t}$.

Answer to **(8)**: $y(t) = c_1 e^{4t} + c_2 e^{6t} - \frac{3}{40}(\cos(2t) + \sin(2t))$.

Review for Chapter 3 - Hints and Solutions

Answer to (1): $Y(s) = \frac{-5(s+3) - 19}{(s+3)^2 + 1}$, $y(t) = -5e^{-3t} \cos(t) - 19e^{-3t} \sin(t)$.

Answer to (2): $y(t) = 5u(t-3) \cdot (t-3)e^{4(t-3)}$.

Answer to (3): $H(s) = \frac{1}{(s-2)^2 + 4^2}$, $h(t) = e^{2t} \frac{\sin(4t)}{4}$.

Answer to (4): $\mathcal{L}[f](s) = e^{-3s} \left(\frac{2}{s^3} - \frac{2}{s} \right)$.

Answer to (5): $Y(s) = \frac{5e^{-3s}}{(s+3)(s^2 - 7s + 12)}$, $y(t) = 5u(t-3) \left(-\frac{1}{6}e^{3(t-3)} + \frac{1}{42}e^{-3(t-3)} + \frac{1}{7}e^{4(t-3)} \right)$.

Answer to (6): $y(t) = -5u(t-9) \left(\frac{1}{4} + \frac{e^{4(t-9)}}{12} - \frac{e^{t-9}}{3} \right)$.

Answer to (7): $y(t) = \frac{e^{25}}{5} u(t-5)(e^{6(t-5)} - e^{t-5})$.

Review for Chapter 4 - Hints and Solutions

Answer to (1):

(a)

$$H'(t) = r_h H \left(1 - \frac{H}{K_h} \right) - \beta HM,$$

$$M'(t) = r_m M \left(1 - \frac{M}{K_m} \right) - \alpha HM.$$

(b) $K_h < K_m$, $r_h < r_m$, $\alpha > \beta$.

Answer to (2):

$$M'(t) = \alpha M - \beta MS,$$

$$S'(t) = -\gamma S + \varepsilon MS - \kappa SH,$$

$$H'(t) = -\lambda H + \mu HS.$$

Answer to (3): (a) $\mathbf{F}(x, y) = \langle -y, -x \rangle$.

(b)

$$\mathbf{F}(1, 1) = \langle -1, -1 \rangle, \quad \mathbf{F}(-1, 1) = \langle -1, 1 \rangle, \quad \mathbf{F}(1, -1) = \langle 1, -1 \rangle, \quad \mathbf{F}(-1, -1) = \langle 1, 1 \rangle.$$

(c) To sketch the direction field it is useful to also plot \mathbf{F} at the following points:

$$\mathbf{F}(1, 0) = \langle 0, -1 \rangle, \quad \mathbf{F}(0, 1) = \langle -1, 0 \rangle, \quad \mathbf{F}(-1, 0) = \langle 0, 1 \rangle, \quad \mathbf{F}(0, -1) = \langle 1, 0 \rangle.$$

It also helps to write the system as $\mathbf{x}' = A \mathbf{x}$, with $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. This matrix A has eigenvalues $\lambda_{\pm} = \pm i$.

(d) The solution curve belongs to the first and second quadrants, starts at $(0, 1)$ for $t = 0$, and notice that

$$\lim_{t \rightarrow \infty} \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad \lim_{t \rightarrow -\infty} \frac{\mathbf{x}(t)}{\|\mathbf{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Review for Chapter 5 - Hints and Solutions

Answer to (1): (a) \mathbf{v}_1 and \mathbf{v}_2 are linearly independent, so the solution exists and it is unique.

(b)

$$\begin{aligned} -x_1 + 3x_2 &= 7, \\ x_1 + 2x_2 &= -3. \end{aligned}$$

The two lines have different slopes, so they intersect at a single point, which means that the solution exists and it is unique.

(c)

$$\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Answer to (2): The vectors \mathbf{v}_1 and \mathbf{v}_2 are parallel, so any linear combination of those two vectors will be a vector collinear with them. So, in order for the system

$$\mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 = \mathbf{c}$$

to have a solution, we need $h = 5/2$. And in this case, the solution is **not** unique.

Answer to (3): About Set S_A :

(a) S_A is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$

(b) Does not apply.

(c) The span of the three vectors is a plane.

(d) A basis for $\text{Span}(S_A)$ is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}.$

About Set S_B :

(a) The set S_B is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} = - \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}.$

(b) Does not apply.

(c) The span of the three vectors is a line.

(d) A basis for $\text{Span}(S_B)$ is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}.$

About Set S_C :

(a) Does not apply.

(b) The set S_C is linearly independent:

$$a \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix} + c \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = b = c = 0.$$

(c) The span of the three vectors is the whole space.

(d) A basis for $\text{Span}(S_C)$ is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$.

Answer to (4): $X = \begin{bmatrix} -31 & 95 \\ \frac{15}{2} & -23 \end{bmatrix}$.

Answer to (5):

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}, \quad e^{At} = \begin{bmatrix} (2 - e^{2t}) & (-1 + e^{2t}) \\ (2 - 2e^{2t}) & (-1 + 2e^{2t}) \end{bmatrix}.$$

Review for Chapter 6 - Hints and Solutions

Answer to (1): (a) $\mathbf{x}(t) = 7e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} - 9e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

(b) $\mathbf{x}(t) = e^{2t} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} + 4e^{2t} \begin{bmatrix} \sin(t) \\ -\cos(t) \end{bmatrix}$.

(c) $\mathbf{x}(t) = 3e^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 8e^{3t} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$.

Answer to (2): $A = \begin{bmatrix} 0 & 1 \\ -5 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ -4 \cos(5t) \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

Answer to (3): (a) (i) $\lambda_{1,2} = 1 \pm 2i$, $\mathbf{v}_{1,2} = \begin{bmatrix} \mp i \\ 2 \end{bmatrix}$.

(ii) $\mathbf{x}_1(t) = e^t \begin{bmatrix} \sin(2t) \\ 2 \cos(2t) \end{bmatrix}$, $\mathbf{x}_2(t) = e^t \begin{bmatrix} -\cos(2t) \\ 2 \sin(2t) \end{bmatrix}$.

(iii) $\mathbf{x}(t) = e^t \begin{bmatrix} \sin(2t) + \cos(2t) \\ 2 \cos(2t) - 2 \sin(2t) \end{bmatrix}$.

(iv) Unstable spiral.

(b) (i) $\lambda_1 = -2$, $\lambda_2 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(ii) $\mathbf{x}_1(t) = e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2(t) = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(iii) $\mathbf{x}(t) = e^{-2t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(iv) Saddle.

(c) (i) $\lambda_1 = -3$, $\lambda_2 = -1$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(ii) $\mathbf{x}_1(t) = e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\mathbf{x}_2(t) = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(iii) $\mathbf{x}(t) = 3e^{-3t} \begin{bmatrix} -1 \\ 2 \end{bmatrix} - 4e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3e^{-3t} + 4e^{-t} \\ 6e^{-3t} - 4e^{-t} \end{bmatrix}$.

(iv) Stable node.

Answer to (4):

$$\mathbf{x}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 4 \\ -9 & 0 \end{bmatrix}, \quad \mathbf{x}_0 \text{ is a center (stable).}$$

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{4}{9} \\ 2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & -8 \\ -9 & -4 \end{bmatrix}, \quad \mathbf{x}_1 \text{ is a saddle (unstable).}$$

$$\mathbf{x}_2 = \begin{bmatrix} -\frac{4}{9} \\ -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & -8 \\ -9 & 4 \end{bmatrix}, \quad \mathbf{x}_2 \text{ is a saddle (unstable).}$$

Review for Chapter 7 - Hints and Solutions

Answer to **(1)**: (a) $y(x) = 2 \cos(3x) - \sin(3x)$

(b) $y(x) = 2 \cos(3x) + k \sin(3x)$ for any $k \in \mathbb{R}$ is a solution

(c) no solutions

Answer to **(2)**:

$$\lambda_n = \left(\frac{(2n-1)\pi}{8}\right)^2, \quad y_n(x) = \sin\left(\frac{(2n-1)\pi}{8}x\right).$$

Answer to **(3)**:

$$f_F(x) = 5 + \sum_{n=1}^{\infty} \left(-12 \frac{\cos(n\pi)}{n\pi}\right) \sin\left(\frac{n\pi x}{3}\right).$$

Answer to **(4)**:

$$f_{eF}(x) = 4 + \sum_{n=1}^{\infty} \frac{12((-1)^n - 1)}{n^2\pi^2} \cos\left(\frac{n\pi x}{3}\right).$$

Answer to **(5)**:

$$v_n(t) = e^{-3\left(\frac{n\pi}{3}\right)^2 t}, \quad w_n(x) = \cos\left(\frac{n\pi x}{3}\right).$$

Answer to **(6)**:

$$u(t, x) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi/4) - \cos(3n\pi/4)}{n} e^{-n^2\pi^2 t/1600} \sin\left(\frac{n\pi x}{40}\right).$$