Review Problems for Exams

Review for Chapter 1 - Hints and Solutions

Answer to **(1)**: **(a)** $P'(t) = \dfrac{1}{5}\, P(t) - 10,$

(b) $P(t) = -10\, e^{t/5} + 50,$

(c) $t_1 = 5\ln(5).$

Answer to **(2)**: (c): $P(t) = 500 + 100\, e^{t/5}.$

Answer to **(3)**: Partial Answer: $y_3(t) = 4 + 8t^4 + 8t^8 + \dfrac{16}{3} t^{12}.$

Answer to **(4)**: $y(t) = \Big(-8\sin(t) + \dfrac{7}{4}\cos(4t) + c\Big)^{-1}$

Answer to **(5)**: (c): $P(t) = 3 + 67\, e^{-2t}.$

Answer to **(6)**: $y(t) = \Big(\dfrac{16}{7}\sin(7t) + \dfrac{1}{4}\Big)^{-1/2}$

Answer to **(7)**: $y(t) = \dfrac{3}{4} t^2 \big(1 - \sin(4t)\big)$

Answer to **(8)**: Part of (b): critical pts: -2 (unstable), 0 (stable) , 2 (unstable).

Answer to **(9)**: $y(t) = \dfrac{8}{\cos(t)} - 5t\tan(t) - 5$

Answer to **(10)**: $y(t) = t\left(\dfrac{17}{2} t^6 - \dfrac{1}{2}\right)^{1/3}.$

Answer to **(11)**: **(a)** $T' = -k(T - 30);$

(b) $T(t) = 30 - 28e^{-kt};$

(c) $k = \dfrac{1}{40}\ln\Big(\dfrac{7}{5}\Big).$

Review for Chapter 2 - Hints and Solutions

Answer to **(1):** $E = (y')^2 + \frac{3}{2}y^2$; $y_{max} = \sqrt{15}$.

Answer to **(2):** $y(t) = c_1 e^{4t} + c_2 t\, e^{4t}$.

Answer to **(3):** $y(t) = \dfrac{8}{3}e^{4t} - \dfrac{23}{3}e^{t}$.

Answer to **(4):** $y(t) = -2e^{4t}\cos(4t) + e^{4t}\sin(4t)$.

Answer to **(5):** $y(t) = -\dfrac{13}{2}e^{5t} + 11e^{3t} + \dfrac{4}{8}e^{t}$.

Answer to **(6):** $y(t) = c_1 e^{4t} + c_2 e^{2t} - \dfrac{3}{2}\, t\, e^{2t}$.

Answer to **(7):** $y(t) = c_1 e^{3t} + c_2 t\, e^{3t} + 2\, t^2\, e^{3t}$.

Answer to **(8):** $y(t) = c_1 e^{4t} + c_2 e^{6t} - \dfrac{3}{40}(\cos(2t) + \sin(2t)))$.

Review for Chapter 3 - Hints and Solutions

Answer to **(1):** $Y(s) = \dfrac{-5(s+3) - 19}{(s+3)^2 + 1}, \quad y(t) = -5e^{-3t}\cos(t) - 19e^{-3t}\sin(t).$

Answer to **(2):** $y(t) = 5u(t-3) \cdot (t-3)e^{4(t-3)}.$

Answer to **(3):** $\mathcal{L}[f](s) = e^{-3s}\left(\dfrac{2}{s^3} - \dfrac{2}{s}\right).$

Answer to **(4):** $Y(s) = \dfrac{5e^{-3s}}{(s+3)(s^2 - 7s + 12)}, \quad y(t) = 5u(t-3)\left(-\dfrac{1}{6}e^{3(t-3)} + \dfrac{1}{42}e^{-3(t-3)} + \dfrac{1}{7}e^{4(t-3)}\right).$

Answer to **(5):** $y(t) = -5u(t-9)\left(\dfrac{1}{4} + \dfrac{e^{4(t-9)}}{12} - \dfrac{e^{t-9}}{3}\right).$

Answer to **(6):** $y(t) = \dfrac{e^{25}}{5}u(t-5)(e^{6(t-5)} - e^{t-5}).$

Review for Chapter 4 - Hints and Solutions

Answer to **(1):**

(a)

$$H'(t) = r_h\, H\Big(1 - \frac{H}{K_h}\Big) - \beta\, HM,$$
$$M'(t) = r_m\, M\Big(1 - \frac{M}{K_m}\Big) - \alpha\, HM.$$

(b) $K_h < K_m, \qquad r_h < r_m, \qquad \alpha > \beta.$

Answer to **(2):**

$$\begin{aligned} M'(t) &= \alpha M - \beta MS, \\ S'(t) &= -\gamma S + \varepsilon MS - \kappa SH, \\ H'(t) &= -\lambda H + \mu HS. \end{aligned}$$

Answer to **(3):** **(a)** $\boldsymbol{F}(x,y) = \langle -y, -x\rangle.$

(b)

$$\boldsymbol{F}(1,1) = \langle -1,-1\rangle, \quad \boldsymbol{F}(-1,1) = \langle -1,1\rangle, \quad \boldsymbol{F}(1,-1) = \langle 1,-1\rangle, \quad \boldsymbol{F}(-1,-1) = \langle 1,1\rangle.$$

(c) To sketch the direction field it is useful to also plot $\boldsymbol{F}$ at the following points:

$$\boldsymbol{F}(1,0) = \langle 0,-1\rangle, \quad \boldsymbol{F}(0,1) = \langle -1,0\rangle, \quad \boldsymbol{F}(-1,0) = \langle 0,1\rangle, \quad \boldsymbol{F}(0,-1) = \langle 1,0\rangle.$$

It also helps to write the system as $\boldsymbol{x}' = A\,\boldsymbol{x}$, with $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. This matrix A has eigenvalues $\lambda_{\pm} = \pm 1$.

(d) The solution curve belongs to the first and second quadrants, starts at $(0, 1)$ for $t = 0$, and notice that

$$\lim_{t\to\infty} \frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lim_{t\to-\infty} \frac{\boldsymbol{x}(t)}{\|\boldsymbol{x}(t)\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Review for Chapter 5 - Hints and Solutions

Answer to **(1)**: **(a)** $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are linearly independent, so the solution exists and it is unique.

(b)

$$-x_1 + 3x_2 = 7,$$
$$x_1 + 2x_2 = -3.$$

The two lines have different slopes, so they intersect at a single point, which means that the solution exists and it is unique.

(c)

$$\begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix} \implies \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

Answer to **(2)**: The vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are parallel, so any linear combination of those two vectors will be a vector collinear with them. So, in order for the system

$$\boldsymbol{v}_1\, x_1 + \boldsymbol{v}_2\, x_2 = \boldsymbol{c}$$

to have a solution, we need $h = 5/2$. And in this case, the solution is **not** unique.

Answer to **(3)**: About Set S_A:

(a) S_A is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(b) Does not apply.

(c) The span of the three vectors is a plane.

(d) A basis for Span(S_A) is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$.

About Set S_B:

(a) The set S_B is linearly dependent: $\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} = - \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$.

(b) Does not apply.

(c) The span of the three vectors is a line.

(d) A basis for Span(S_B) is $\left\{ \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix} \right\}$.

About Set S_C:

(a) Does not apply.

(b) The set S_C is linearly independent:

$$a\begin{bmatrix}-1\\1\\0\end{bmatrix}+b\begin{bmatrix}0\\3\\2\end{bmatrix}+c\begin{bmatrix}2\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\Rightarrow\quad a=b=c=0.$$

(c) The span of the three vectors is a the whole space.

(d) A basis for $\text{Span}(S_C)$ is $\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\3\\2\end{bmatrix},\begin{bmatrix}2\\1\\-1\end{bmatrix}\right\}$.

Answer to **(4):** $X=\begin{bmatrix}-31 & 95\\ \frac{15}{2} & -23\end{bmatrix}$.

Answer to **(5):**

$$A=\begin{bmatrix}1 & 1\\2 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}-1 & 1\\2 & -1\end{bmatrix},\qquad e^{At}=\begin{bmatrix}(2-e^{2t}) & (-1+e^{2t})\\(2-2e^{2t}) & (-1+2e^{2t})\end{bmatrix}.$$

Review for Chapter 6 - Hints and Solutions

Answer to **(1)**: **(a)** $\boldsymbol{x}(t) = 7\,e^{3t}\begin{bmatrix}1\\-2\end{bmatrix} - 9\,e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}.$

(b) $\boldsymbol{x}(t) = e^{2t}\begin{bmatrix}\cos(t)\\ \sin(t)\end{bmatrix} + 4\,e^{2t}\begin{bmatrix}\sin(t)\\-\cos(t)\end{bmatrix}.$

(c) $\boldsymbol{x}(t) = 3\,e^{3t}\begin{bmatrix}1\\-1\end{bmatrix} - 8\,e^{3t}\Big(\begin{bmatrix}1\\-1\end{bmatrix}t + \begin{bmatrix}1\\0\end{bmatrix}\Big).$

Answer to **(2)**: $A = \begin{bmatrix}0 & 1\\-5 & -1\end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix}0\\-4\cos(5t)\end{bmatrix}, \qquad \boldsymbol{x}(0) = \begin{bmatrix}-3\\2\end{bmatrix}.$

Answer to **(3)**: **(a)** **(i)** $\lambda_{1,2} = 1 \pm 2i,\ \boldsymbol{v}_{1,2} = \begin{bmatrix}\mp i\\2\end{bmatrix}.$

(ii) $\boldsymbol{x}_1(t) = e^t\begin{bmatrix}\sin(2t)\\2\cos(2t)\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^t\begin{bmatrix}-\cos(2t)\\2\sin(2t)\end{bmatrix}.$

(iii) $\boldsymbol{x}(t) = e^t\begin{bmatrix}\sin(2t)+\cos(2t)\\2\cos(2t)-2\sin(2t)\end{bmatrix}.$

(iv) Unstable spiral.

(b) **(i)** $\lambda_1 = -2, \qquad \lambda_2 = 1, \qquad \boldsymbol{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}.$

(ii) $\boldsymbol{x}_1(t) = e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^{t}\begin{bmatrix}1\\1\end{bmatrix}.$

(iii) $\boldsymbol{x}(t) = e^{-2t}\begin{bmatrix}1\\2\end{bmatrix}.$

(iv) Saddle.

(c) **(i)** $\lambda_1 = -3, \qquad \lambda_2 = -1, \qquad \boldsymbol{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix}, \qquad \boldsymbol{v}_2 = \begin{bmatrix}-1\\1\end{bmatrix}.$

(ii) $\boldsymbol{x}_1(t) = e^{-3t}\begin{bmatrix}-1\\2\end{bmatrix}, \qquad \boldsymbol{x}_2(t) = e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}.$

(iii) $\boldsymbol{x}(t) = 3\,e^{-3t}\begin{bmatrix}-1\\2\end{bmatrix} - 4\,e^{-t}\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-3\,e^{-3t}+4\,e^{-t}\\6\,e^{-3t}-4\,e^{-t}\end{bmatrix}.$

(iv) Stable node.

Answer to **(4)**:

$$\mathbf{x}_0 = \begin{bmatrix}0\\0\end{bmatrix}, \qquad A_0 = \begin{bmatrix}0 & 4\\-9 & 0\end{bmatrix}, \qquad \mathbf{x}_0 \quad \text{is a center (stable).}$$

$$\mathbf{x}_1 = \begin{bmatrix} -\frac{4}{9} \\ 2 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} 0 & -8 \\ -9 & -4 \end{bmatrix}, \qquad \mathbf{x}_1 \quad \text{is a saddle (unstable).}$$

$$\mathbf{x}_2 = \begin{bmatrix} -\frac{4}{9} \\ -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & -8 \\ -9 & 4 \end{bmatrix}, \qquad \mathbf{x}_2 \quad \text{is a saddle (unstable).}$$

Review for Chapter 7 - Hints and Solutions

Answer to **(1):** (a) $y(x) = 2\cos(3x) - \sin(3x)$

(b) $y(x) = 2\cos(3x) + k\sin(3x)$ for any $k \in \mathbb{R}$ is a solution

(c) no solutions

Answer to **(2):**

$$\lambda_n = \Big(\frac{(2n-1)\pi}{8}\Big)^2, \qquad y_n(x) = \sin\Big(\frac{(2n-1)\pi}{8}x\Big).$$

Answer to **(3):**

$$f_F(x) = 5 + \sum_{n=1}^{\infty} \left(-12\frac{\cos(n\pi)}{n\pi}\right) \sin\left(\frac{n\pi x}{3}\right).$$

Answer to **(4):**

$$f_{eF}(x) = 4 + \sum_{n=1}^{\infty} \frac{12((-1)^n - 1)}{n^2\pi^2} \cos(\frac{n\pi x}{3}).$$

Answer to **(5):**

$$v_n(t) = e^{-3\left(\frac{n\pi}{3}\right)^2 t}, \qquad w_n(x) = \cos\left(\frac{n\pi x}{3}\right).$$

Answer to **(6):**

$$u(t,x) = \frac{100}{\pi} \sum_{n=1}^{\infty} \frac{\cos(n\pi/4) - \cos(3n\pi/4)}{n} e^{-n^2\pi^2 t/1600} \sin\Big(\frac{n\pi x}{40}\Big).$$