Linear Systems of Differential Equations

Two-Dimensional linear systems of differential equations

Objectives

To better understand the solutions of 2×2 linear systems of first order differential equations having constant coefficients and no sources.

Recitation Worksheet Problems: Sections 6.1, 6.2

For each of the matrices **(a)-(d)** below do the following:

(1) Find the eigenvalues and eigenvectors of the matrix A.

(2) Determine the type of equilibrium of the trivial solution and its stability:

- stable/unstable node,
- stable/unstable spiral,
- center,
- saddle.

(3) Find a set of fundamental solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

(4) Find the particular solution satisfying $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(5) Only for cases **(a)-(c)** sketch a phase portrait of the solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.
Note: Make sure your phase portrait captures all qualitatively different solutions.

The matrices are:

$$(\mathbf{a}) \quad A = \begin{bmatrix} -3 & 4 \\ -1 & -3 \end{bmatrix}, \quad (\mathbf{b}) \quad A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}, \quad (\mathbf{c}) \quad A = \begin{bmatrix} 7 & -2 \\ -4 & 5 \end{bmatrix}, \quad (\mathbf{d}) \quad A = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}.$$

Solutions for the Eigenpairs

The eigenvalues for matrices in **(a)-(d)** above are:

$$(\mathbf{a})\quad \lambda_{\pm} = -3 \pm 2i, \quad (\mathbf{b})\quad \begin{cases} \lambda_{+} = 4 \\ \lambda_{-} = -1. \end{cases} \quad (\mathbf{c})\quad \begin{cases} \lambda_{+} = 9 \\ \lambda_{-} = 3. \end{cases} \quad (\mathbf{d})\quad \lambda_{\pm} = -3.$$

The eigenvectors for matrices in **(a)-(d)** above are:

$$(\mathbf{a})\quad \boldsymbol{v}_{\pm} = \begin{bmatrix} 2 \\ \pm i \end{bmatrix}, \quad (\mathbf{b})\quad \boldsymbol{v}_{+} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_{-} = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$(\mathbf{c})\quad \boldsymbol{v}_{+} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_{-} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (\mathbf{d})\quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$