Matrix Algebra and Differential systems

Overview of matrix algebra and linear systems of differential equations

Objectives

To better understand matrix functions, exponentials, and solutions of linear differential system.

Recitation Worksheet Problems: Sections 5.5, 5.6, 6.1

1. Consider the matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$.

(1a) Find the matrix function $F(t) = e^{At}$, for $t \in \mathbb{R}$.

(1b) Verify both that $F'(t) = A\,F(t)$ and $F'(t) = F(t)\,A$.

2. Show that the matrix A below is diagonalizable and write $A = PDP^{-1}$ in two different ways, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

3. Consider the following matrices:

$$A_0 = \begin{bmatrix} -7 & 6 \\ -4 & 7 \end{bmatrix}, \qquad B_0 = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \qquad C_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(3a) Find the eigenpairs for these matrices, and determine which of them are diagonalizable.

(3b) Find all solutions of the initial value problem $\boldsymbol{x}'(t) = A_0\,\boldsymbol{x}(t)$ with A_0 given above.

(3c) Find the solution of the initial value problem

$$\boldsymbol{x}'(t) = A_0\,\boldsymbol{x}(t), \qquad \boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1.

$$e^{At} = \frac{1}{5}\begin{bmatrix}(4\,e^{3t} + e^{-2t}) & (4\,e^{3t} - 4\,e^{-2t})\\ (e^{3t} - e^{-2t}) & (e^{3t} + 4\,e^{-2t})\end{bmatrix}.$$

2. Eigenpairs:

$$\lambda = 2, \quad \boldsymbol{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \quad \text{and} \quad \lambda = 3, \quad \boldsymbol{v}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}.$$

3. (a)

$$\boldsymbol{x}(t) = \frac{2}{5}\,e^{5t}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{5}\,e^{-5t}\begin{bmatrix}3\\1\end{bmatrix}.$$

(b)

$$\boldsymbol{x}(t) = e^{3t}\begin{bmatrix}1+t\\1\end{bmatrix}.$$

(c)

$$\boldsymbol{x}(t) = \begin{bmatrix}\cos(t) + \sin(t)\\ -\sin(t) + \cos(t)\end{bmatrix}.$$