Vector Spaces

Overview of vector spaces, subspaces, and linear (in)dependence

Objectives

To better understand vector spaces, subspaces, linear (in)dependence of vectors, basis of a vector space.

Recitation Worksheet Problems: Sections 5.1, 5.2.

1. Sketch a graph of the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

(a) Does the linear system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{b}$ have a solution? Determine if the solution is unique, based on the "column picture", i.e. based on linear dependence/independence of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

(b) Write the above system in the form of two (scalar) equations and interpret the existence (or not) of a solution in terms of lines intersecting (or not).

(c) *(If in lecture you got to Sect. 5.3)* Write the above system in matrix form. Determine if the matrix is invertible. If so, find the solution using the formula for matrix inverse.

2. Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix},$$

and given a real number h, define $\boldsymbol{c} = \left[\begin{smallmatrix} 2 \\ h \end{smallmatrix}\right]$. Plot the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and find all values of h such that the system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{c}$ has a solution. Determine if the solution is unique.

3. Determine which of the sets of vectors below is linearly independent, and also answer the following:

(i) If the set is linearly dependent, express one vector as a non-zero linear combination of the other vectors in the set.

(ii) If the set is linearly independent, show that the only linear combination of the above vectors which gives the zero vector is such that all scalars are zero.

(iii) For each of the sets, determine if the span of the vectors is the whole space, a plane, or a line.

(iv) For each of the sets, find a basis for their span.

(A) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; **(B)** $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(C) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix} \right\}$; **(D)** $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$; **(E)** $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix} \right\}$.

Answer Key.

1. (a) Yes, because $\boldsymbol{v}_1$ is not parallel to $\boldsymbol{v}_2$. And for this reason the solution is unique, as can be seen in a picture. The solution is $x_1 = -3$, $x_2 = 2$.

(b)

$$\begin{aligned} x_1 &= -3 \\ x_1 + 2x_2 &= 1. \end{aligned}$$

The two lines intersect.

(c)

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

2. Since $\boldsymbol{v}_1$ is parallel to $\boldsymbol{v}_2$, the system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{c}$ has solutions only if $\boldsymbol{c}$ is also parallel to $\boldsymbol{v}_1$. This condition means

$$\begin{bmatrix} 2 \\ h \end{bmatrix} = k \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \Rightarrow \quad k = 2, \quad h = -3\,k \quad \Rightarrow \quad h = -6.$$

The solution of the system $\begin{bmatrix} 1 \\ -3 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 9 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ has infinitely many solutions, since the coefficient vectors are linearly dependent.

3. **(A)** LI
(B) $\boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{0}$
(C) $3\boldsymbol{v}_1 - 9\boldsymbol{v}_2 + \boldsymbol{v}_3 = \boldsymbol{0}$
(D) LI
(E) $\boldsymbol{v}_2 = -2\boldsymbol{v}_1$ and $\boldsymbol{v}_3 = 3\boldsymbol{v}_1$