Introduction to Modeling

Construction of Models and Description of their Solutions

Objectives

- Students should be able to create mathematical models from a given description of a physical sitation.
- Conversely, given the differential equations (for example, of a population model), students should know the meaning of the equation coefficients and then be able to describe what type of population system is described by these equations.
- Students should be able to obtain a qualitative description of the solutions of autonomous differential equations.

Recitation Worksheet Problems: Sections 1.2, 1.3, 1.4

1. A population of fish has a **growth rate proportional to the amount of fish present** at that time, with a proportionality factor of $\frac{1}{3}$ per unit time.

   (a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the number of fish as a function of time.

   (b) Now, assume that we have the same fish population, reproducing as above, but we are harvesting fish at a **constant rate of 300** fish per unit time. Write the differential equation in this case.

   (c) Assume that the initial fish population is 500 fish. Solve the ordinary differential equation in part (b) above with this given initial condition.

2. A radioactive material has a **decay rate** proportional to the amount of radioactive material present at that time, with a proportionality factor of 5 per unit time.

   (a) Write a differential equation of the form $P' = F(P)$, which models this situation, where P is the amount of radioactive material micrograms as function of time.

   (b) Now, assume that we have the same radioactive material decaying as above, but we are **adding** additional material (of the same type) **at a constant rate of 3** micrograms per unit time. Write the differential equation in this case.

   (c) Solve the ordinary differential equation in part (b) above, assuming the initial amount of radioactive material is 10 micrograms.

3. Consider the population model

$$\frac{dP}{dt} = -3\,(P-5)\,(P-10)\,(P-20)\,,$$

where $P(t)$ is the population at time t.

(a) Determine the **equilibria** of the differential equation.

(b) Determine the intervals where the population is **increasing** and where it is **decreasing**.

(c) Determine the stability of the equilibria.

4. Consider two populations of organism, which occupy the same environment. Let $A(t)$ and $B(t)$ denote the number of organisms at time t of the first and second kind, respectively. They are modeled by the following system of equations

$$A' = 2A - \frac{A^2}{10} - 3AB$$
$$B' = 3B - \frac{B^2}{5} + 2AB.$$

(a) Determine the growth rate coefficients of A and B (when the organisms are not sensing that resources are limited).

(b) Determine the carrying capacity of each of the populations (if the other population didn't exist).

(c) Determine the interaction between the two populations: are they cooperating, are they competing? If they have different types of behaviors, which one is helping and which one is hurting the other?

5. Consider the initial value problem

$$y'(t) = \frac{t}{y(t) + t^2\,y(t)}, \qquad y(0) = 1.$$

(a) Find an **implicit** expression of all solutions, $y(t)$, of the differential equation above, in the form $\psi(t, y) = c$, where c collects all constant terms.

(b) Find the **explicit** expression of the solution $y(t)$ of the initial value problem above.