Linear Systems of Differential Equations

Two-Dimensional linear systems of differential equations

Objectives

To understand the solutions of 2×2 linear systems of first order differential equations having constant coefficients and no sources.

Recitation Worksheet Problems: Sections 6.1, 6.2

Consider the matrices

$$(\mathbf{a}) \quad A = \begin{bmatrix} -3 & 4 \\ -1 & -3 \end{bmatrix}, \quad (\mathbf{b}) \quad A = \begin{bmatrix} 3 & 4 \\ 1 & 0 \end{bmatrix}, \quad (\mathbf{c}) \quad A = \begin{bmatrix} 7 & -2 \\ -4 & 5 \end{bmatrix}, \quad (\mathbf{d}) \quad A = \begin{bmatrix} -2 & -1 \\ 1 & -4 \end{bmatrix}.$$

For each of the matrices **(a)-(d)** above do the following:

(1) Find the eigenvalues and eigenvectors of the matrices A.

The eigenvalues for matrices in **(a)-(d)** above are:

$$(\mathbf{a}) \quad \lambda_\pm = -3 \pm 2i, \quad (\mathbf{b}) \quad \begin{cases} \lambda_+ = 4 \\ \lambda_- = -1. \end{cases} \quad (\mathbf{c}) \quad \begin{cases} \lambda_+ = 9 \\ \lambda_- = 3. \end{cases} \quad (\mathbf{d}) \quad \lambda_\pm = -3.$$

The eigenvectors for matrices in **(a)-(d)** above are:

$$(\mathbf{a}) \quad \boldsymbol{v}_\pm = \begin{bmatrix} 2 \\ \pm i \end{bmatrix}, \quad (\mathbf{b}) \quad \boldsymbol{v}_+ = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

$$(\mathbf{c}) \quad \boldsymbol{v}_+ = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \boldsymbol{v}_- = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad (\mathbf{d}) \quad \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

(2) Classify the trivial solution $\boldsymbol{x} = \mathbf{0}$ as one of the following: Source Node, Sink Node, Saddle Node, Source Spiral, Sink Spiral, Center.

(a) Sink Spiral.

(b) Saddle Node.

(c) Source Node.

(d) Sink Node.

(3) Find a set of real fundamental solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

(a) $\boldsymbol{x}_1(t) = e^{-3t} \begin{bmatrix} 2\cos(2t) \\ -\sin(2t) \end{bmatrix}$ and $\boldsymbol{x}_2(t) = e^{-3t} \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \end{bmatrix}$.

(b) $\boldsymbol{x}_+(t) = e^{4t}\begin{bmatrix}4\\1\end{bmatrix}$ and $\boldsymbol{x}_-(t) = e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$.

(c) $\boldsymbol{x}_+(t) = e^{9t}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\boldsymbol{x}_-(t) = e^{3t}\begin{bmatrix}1\\2\end{bmatrix}$.

(d) $\boldsymbol{x}_1(t) = e^{-3t}\begin{bmatrix}1\\1\end{bmatrix}$ and $\boldsymbol{x}_2(t) = e^{-3t}\Big(\begin{bmatrix}1\\1\end{bmatrix}t + \begin{bmatrix}1\\0\end{bmatrix}\Big)$.

(4) Find the particular solution satisfying $\boldsymbol{x}(0) = \begin{bmatrix}1\\2\end{bmatrix}$.

(a) $\boldsymbol{x}(t) = e^{-3t}\Big(\frac{1}{2}\begin{bmatrix}2\cos(2t)\\-\sin(2t)\end{bmatrix} + 2\begin{bmatrix}2\sin(2t)\\\cos(2t)\end{bmatrix}\Big)$.

(b) $\boldsymbol{x}(t) = \frac{3}{5}e^{4t}\begin{bmatrix}4\\1\end{bmatrix} + \frac{7}{5}e^{-t}\begin{bmatrix}-1\\1\end{bmatrix}$.

(c) $\boldsymbol{x}(t) = e^{3t}\begin{bmatrix}1\\2\end{bmatrix}$.

(d) $\boldsymbol{x}(t) = e^{-3t}\Big((-t+2)\begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}1\\0\end{bmatrix}\Big)$.

(5) Only for cases **(a)-(c)** sketch a phase portrait of the solutions of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$.

Note: Make sure your phase portrait captures all qualitatively different solutions.