# Second Order Equations

*Solving second order linear homogeneous differential equations*

## Objectives

Students should know the general properties of solutions to second order linear differential equations (SOLDE). In the case the SOLDE is Newton's law of motion, students should be able to get information about their solutions using the conservation of the mechanical energy.

Students should be able to SOLDE with constant coefficients. They should know how to write the general solutions of these equations in the case that the characteristic polynomial of the equation has real-valued, complex-valued, or repeated roots. In the case of solutions that oscillate in time, students should be able to find real-valued expressions of these solutions, and also write them in terms of amplitude and phase shift.

## Recitation Worksheet Problems: Sections 2.1, 2.2

**(1)** A mass-spring system with mass $m > 0$ and spring constant $k > 0$ is oscillating in a medium without friction. The initial displacement from equilibrium is $y(0) = y_0$, positive downwards, and the initial velocity is $y'(0) = v_0$. Find formulas for the maximum velocity and maximum displacement achieved during the motion of this system.

***(10 minutes)*** *Newton's equation of motion is*

$$m\,y'' + k\,y = 0 \quad \Rightarrow \quad \Big(m\,y'' + k\,y\Big)\,y' = 0 \quad \overset{\text{chain rule}}{\Longleftrightarrow} \quad \Big(\frac{1}{2}m\,v^2(t) + \frac{1}{2}k\,y^2(t)\Big)' = 0,$$

*where* $v(t) = y'(t)$. *The mechanical energy is* $E(t) = \frac{1}{2}m\,v^2(t) + \frac{1}{2}k\,y^2(t)$. *Newton's equation says,*

$$E'(t) = 0 \quad \Leftrightarrow \quad E(t) = E(0), \qquad \text{(conservation of the mechanical energy).}$$

*The initial conditions fixes the initial value of the energy,* $E(0) = \frac{1}{2}m\,v_0^2 + \frac{1}{2}k\,y_0^2$. *From the conservation of the mechanical energy* $\frac{1}{2}m\,v^2(t) + \frac{1}{2}k\,y^2(t) = E(0)$ *we get* $v_{\max}$ *and* $y_{\max}$, *because*

$$v_{\max} \quad \text{when} \quad y = 0 \quad \Rightarrow \quad \frac{1}{2}m\,v_{\max}^2 + 0 = E(0) \quad \Rightarrow \quad \boxed{v_{\max} = \sqrt{\frac{2E(0)}{m}}},$$

$$y_{\max} \quad \text{when} \quad v = 0 \quad \Rightarrow \quad 0 + \frac{1}{2}k\,y_{\max}^2 = E(0) \quad \Rightarrow \quad \boxed{y_{\max} = \sqrt{\frac{2E(0)}{k}}}.$$

**(2)** If $y_1(t) = e^t$ is solution of $L(y_1) = 0$ and $y_2(t)$ is solution of $L(y_2) = 2\,e^{3t}$, where $L(y) = y'' + 3y' - 4y$, then mark with True or False the following statements.

$$L(y_1 + y_2) = 0, \qquad L(2y_1) = 0, \qquad L((y_1)^2) = 0,$$
$$L(2y_1 - y_2) = -6\,e^{3t}, \qquad L(2y_2) = 2\,e^{3t}, \qquad L(7y_2) = 14\,e^{3t}.$$

***(5 minutes)***

$$L(y_1 + y_2) = 0 \quad F, \qquad L(2y_1) = 0 \quad T, \qquad L((y_1)^2) = 0 \quad F,$$
$$L(2y_1 - y_2) = -2\,e^{3t} \quad T, \qquad L(2y_2) = 2\,e^{3t} \quad F, \qquad L(7y_2) = 14\,e^{3t} \quad T.$$

**(3)** Solve the initial value problem

$$y'' - y' - 2y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

***(5 minutes)*** $y(t) = \dfrac{2}{3}\,e^{2t} + \dfrac{1}{3}\,e^{-t}.$

**(4)** Solve the initial value problem

$$y'' + 10y' + 25y = 0, \qquad y(0) = 1, \qquad y'(0) = 1.$$

***(5 minutes)*** $y(t) = e^{-5t} + 6t\,e^{-5t}.$

**(5)** Solve the initial value problem and write the solution using real-valued expressions only.

$$y'' - 2y' + 5y = 0, \qquad y(0) = -1, \qquad y'(0) = 1.$$

***(10 minutes)*** $y(t) = -e^t\cos(2t) + e^t\sin(2t).$

**(6)** **(Optional Problem)** Write the solution in part (5) in terms of amplitude and phase shift.

***(10 minutes)*** $y(t) = \sqrt{2}\,e^t\cos\Big(2t - \dfrac{3\pi}{4}\Big).$