Diagonalizable Matrices

Diagonalizable matrices and their exponential

Objectives

To better understand matrix functions, exponentials, and solutions of linear differential system.

Solutions to Recitation Worksheet Problems: Sections 5.4, 5.5, 5.6

(1) (1a)

$$0 = \begin{vmatrix} (2-\lambda) & 4 \\ 1 & (-1-\lambda) \end{vmatrix} = (2-\lambda)(-1-\lambda) - 4 = \lambda^2 - \lambda - 6$$

$$\lambda_\pm = \frac{1}{2}\big(1 \pm \sqrt{1+24}\big) = \frac{1}{2}(1 \pm 5) = \begin{cases} \lambda_+ = 3 \\ \lambda_- = -2. \end{cases}$$

For $\lambda_+ = 3$ we get

$$(A-3I)\boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} -1 & 4 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = 4v_2 \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 4v_2 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

So first eigenpair: $\lambda_+ = 3$, $\boldsymbol{v}_+ = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

For $\lambda_- = -2$ we get

$$(A+2I)\boldsymbol{v} = \boldsymbol{0} \quad \Rightarrow \quad \begin{bmatrix} 4 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v_1 = -v_2 \quad \Rightarrow \quad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So second eigenpair: $\lambda_- = -2, \quad \boldsymbol{v}_- = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Therefore, we can write

$$A = PDP^{-1}, \qquad P = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, \qquad P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix},$$

Since $At = P(Dt)P^{-1}$, then

$$e^{At} = P(e^{Dt})P^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix},$$

$$e^{At} = \frac{1}{5}\begin{bmatrix} 4\,e^{3t} & -e^{-2t} \\ e^{3t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 4 \end{bmatrix}$$

$$\boxed{e^{At} = \frac{1}{5}\begin{bmatrix} (4\,e^{3t} + e^{-2t}) & (4\,e^{3t} - 4\,e^{-2t}) \\ (e^{3t} - e^{-2t}) & (e^{3t} + 4\,e^{-2t}) \end{bmatrix}}.$$

(1b) Short Version of the Proof:

$$\big(e^{At}\big)' = \big(Pe^{Dt}P^{-1}\big)' = P\big(e^{Dt}\big)'P^{-1} = Pe^{Dt}DP^{-1} = Pe^{Dt}P^{-1}\;PDP^{-1} = e^{At}A.$$
$$= PDe^{Dt}P^{-1} = PDP^{-1}\;Pe^{Dt}P^{-1} = A\,e^{At}.$$

Long Version of the Proof:

$$\big(e^{At}\big)' = \frac{1}{5}\begin{bmatrix}(12\,e^{3t} - 2\,e^{-2t}) & (12\,e^{3t} + 8\,e^{-2t})\\ (3e^{3t} + 2\,e^{-2t}) & (3\,e^{3t} - 8\,e^{-2t})\end{bmatrix}$$

$$\begin{aligned} e^{At}\,A &= \frac{1}{5}\begin{bmatrix}(4\,e^{3t} + e^{-2t}) & (4\,e^{3t} - 4\,e^{-2t})\\ (e^{3t} - e^{-2t}) & (e^{3t} + 4\,e^{-2t})\end{bmatrix}\begin{bmatrix}2 & 4\\ 1 & -1\end{bmatrix}\\ &= \frac{1}{5}\begin{bmatrix}((8+4)\,e^{3t} + (2-4)\,e^{-2t}) & ((16-4)\,e^{3t} + (4+4)\,e^{-2t})\\ ((2+1)\,e^{3t} + (-2+4)\,e^{-2t}) & ((4-1)\,e^{3t} + (-4-4)\,e^{-2t})\end{bmatrix}\\ &= \frac{1}{5}\begin{bmatrix}(12\,e^{3t} - 2\,e^{-2t}) & (12\,e^{3t} + 8\,e^{-2t})\\ (3e^{3t} + 2\,e^{-2t}) & (3\,e^{3t} - 8\,e^{-2t})\end{bmatrix}\\ &= \big(e^{At}\big)'. \end{aligned}$$

We leave $A\,e^{At}$ as exercise.

(2) Eigenpairs:

$$\boxed{\lambda = 2, \quad \boldsymbol{v}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix}}, \quad \text{and} \quad \boxed{\lambda = 3, \quad \boldsymbol{v}_3 = \begin{bmatrix}1\\2\\1\end{bmatrix}}.$$

(3)

- Matrix A is diagonalizable with $D_A = \begin{bmatrix}5 & 0\\ 0 & -5\end{bmatrix}$ and $P_A = \begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}$.
- Matrix B is not diagonalizable, because it has only one eigenvalue $\lambda = 3$ and the corresponding set of eignevectors is only a line in the direction of $\boldsymbol{v} = \begin{bmatrix}1\\0\end{bmatrix}$.
- If matrix C is a function on real vectors only, then C is not diagonalizable, since it does not have any real eigenvalues.
- If matrix C is a function on complex vectors, then C is diagonalizable with $D_C = \begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix}$ and $P_C = \begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$.

(4) Extra Problem from Section 6.1:

(4a)

$$\boxed{\boldsymbol{x}(t) = c_1\begin{bmatrix}4\\1\end{bmatrix}e^{3t} + c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{-2t}}, \qquad c_1, c_2 \in \mathbb{R}.$$

(4b)

$$\boxed{\boldsymbol{x}(t) = \frac{2}{5}\begin{bmatrix}4\\1\end{bmatrix}e^{3t} + \frac{3}{5}\begin{bmatrix}-1\\1\end{bmatrix}e^{-2t}} \iff \boxed{\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) = \frac{1}{5}\begin{bmatrix}(4\,e^{3t} + e^{-2t}) & (4\,e^{3t} - 4\,e^{-2t})\\ (e^{3t} - e^{-2t}) & (e^{3t} + 4\,e^{-2t})\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}.$$