Vector Spaces

Overview of vector spaces, subspaces, and linear (in)dependence

Objectives

To better understand vector spaces, subspaces, linear (in)dependence of vectors, basis of a vector space.

Recitation Worksheet Problems: Sections 5.1, 5.2.

(1) Sketch a graph of the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix},$$

(1a) Does the linear system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{b}$ have a solution? Determine if the solution is unique, based on the "column picture", i.e. based on linear dependence/independence of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

(1b) Write the above system in the form of two (scalar) equations and interpret the existence (or not) of a solution in terms of lines intersecting (or not).

(1c) *(If in lecture you covered Sect. 5.3)* Write the above system in matrix form. Determine if the matrix is invertible. If so, find the solution using the formula for matrix inverse.

(2) Consider the vectors

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix},$$

and given a real number h, define $\boldsymbol{c} = \left[\begin{smallmatrix} 2 \\ h \end{smallmatrix}\right]$. Plot the vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ and find all values of h such that the system $\boldsymbol{v}_1 x_1 + \boldsymbol{v}_2 x_2 = \boldsymbol{c}$ has a solution. Determine if the solution is unique.

(3) Determine which of the sets of vectors below is linearly independent, and also answer the following:

- If the set is linearly dependent, express one vector as a non-zero linear combination of the other vectors in the set.
- If the set is linearly independent, show that the only linear combination of the above vectors which gives the zero vector is such that all scalars are zero.
- For each of the sets, determine if the span of the vectors is the whole space, a plane, or a line.
- For each of the sets, find a basis for their span.

(A) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$; **(B)** $\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$;

(C) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix} \right\}$; **(D)** $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$; **(E)** $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ -9 \\ 6 \end{bmatrix} \right\}$.