Vector Spaces

Overview of vector spaces, subspaces, and linear (in)dependence

Recitation Worksheet Solutions: Sections 5.1, 5.2.

(1) **(1a)** The system has a solution and it is unique, because $\boldsymbol{v}_1$ is not parallel to $\boldsymbol{v}_2$. The solution is

$$x_1 = -3, \qquad x_2 = 2.$$

(1b)

$$\begin{aligned} x_1 + 0\,x_2 &= -3 \\ x_1 + 2x_2 &= 1. \end{aligned}$$

The solutions for each equation form a line, and these lines do not intersect.

(1c)

$$\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

(2) Since $\boldsymbol{v}_1$ is parallel to $\boldsymbol{v}_2$, the system $\boldsymbol{v}_1\,x_1 + \boldsymbol{v}_2\,x_2 = \boldsymbol{c}$ has solutions if and only if $\boldsymbol{c}$ is also parallel to $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. The condition that $\boldsymbol{c}$ is parallel to $\boldsymbol{v}_1$ means

$$\begin{bmatrix} 2 \\ h \end{bmatrix} = k \begin{bmatrix} 1 \\ -3 \end{bmatrix} \quad \Rightarrow \quad k = 2, \quad h = -3\,k \quad \Rightarrow \quad h = -6.$$

The solution of the system $\begin{bmatrix} 1 \\ -3 \end{bmatrix} x_1 + \begin{bmatrix} -3 \\ 9 \end{bmatrix} x_2 = \begin{bmatrix} 2 \\ -6 \end{bmatrix}$ has infinitely many solutions, since the coefficient vectors are linearly dependent.

(3) Let's denote the vectors in the sets as $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$.

(A) Linearly Independent. Their span is the whole space $\mathbb{R}^3$.

(B) $\boldsymbol{v}_1 - \boldsymbol{v}_2 - \boldsymbol{v}_3 = \boldsymbol{0}$. Their span is a plane in $\mathbb{R}^3$.

(C) $3\boldsymbol{v}_1 - 9\boldsymbol{v}_2 + \boldsymbol{v}_3 = \boldsymbol{0}$. Their span is a plane in $\mathbb{R}^3$.

(D) Linearly Independent. Their span is the whole space $\mathbb{R}^3$.

(E) $\boldsymbol{v}_2 = -2\boldsymbol{v}_1$ and $\boldsymbol{v}_3 = 3\boldsymbol{v}_1$. Their span is a line in $\mathbb{R}^3$.