Review of Linear Algebra (Chp 8)

Section Objective(s):

- Why do we need Linear Algebra?
- An Overview of Matrix Algebra.
- Eigenvalues and Eigenvectors of a Matrix.
- Diagonalizable Matrices.
- The Exponential of a Matrix.

Why Do We Need Linear Algebra?

Because we are going to study ____________ of linear differential equations.

Recall: In section 1.1 we found out that,

____.

Now we want to solve ____________ of equations:

____,

____.

We write this ____________ of equations as

Remark: We need to understand what is the ________________ of a matrix.

An Overview of Matrix Algebra.

Definition 8.1.2. An $m \times n$ ***matrix***, A, is an ____________________________

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}, \qquad \begin{array}{c} m \text{ rows,} \\ n \text{ columns,'} \end{array}$$

A ***square matrix*** is an ____________________________.

Remarks:

(a) The 3×3 matrix A below is ____________________________,

and B is ____________________________.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}.$$

(b) The particular case of an __________ matrix is called an m-vector, $\boldsymbol{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix}$.

(c) The ______________________________________ of matrices is defined for matrices

of the same size, ____.

(d) The matrix multiplication is defined for matrices such that the numbers of columns in the first matrix ____________________ the numbers of rows in the second matrix.

Example 8.2.12: Compute AB, where $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}$.

Solution:

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Example 8.2.14: Compute AB and BA, where $A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution:

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EXAMPLE 8.2.15: Compute AB and BA, where $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$.

SOLUTION:

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Definition 8.2.8. I_n is the $n \times n$ ***identity matrix*** iff for every n-vector $\boldsymbol{x}$ holds

__________________________.

Remark: The cases $n = 2, 3$ are given by

.

Definition 8.2.9. A square matrix A is ***invertible*** iff there is a matrix A^{-1} so that

____.

EXAMPLE 8.2.16: Verify that the matrix and its inverse are given by

$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}, \qquad A^{-1} = \frac{1}{4} \begin{bmatrix} 3 & -2 \\ -1 & 2 \end{bmatrix}.$$

SOLUTION:

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Theorem 8.2.10. Given a 2×2 matrix A, let Δ be the number

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}.$$

Then, A is invertible iff $\Delta \neq 0$. Furthermore, if A is invertible, its inverse is

____.

Remarks:

(a) The number Δ is called the ______________________________ of A.

(b) Δ ____________________________ whether A is invertible or not

EXAMPLE 8.2.17: Compute the inverse of matrix $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, given in Example 8.2.15.

SOLUTION:

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Example 8.2.19: Find a matrix X such that $AXB = I$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solution:

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Definition 8.2.11. The ***determinant of a*** 2×2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \rule{10cm}{0.4pt}.$$

Definition 8.2.11. The ***determinant of a*** 3×3 matrix $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

EXAMPLE 8.2.23: Compute the determinant of the 3×3 matrix,

$$\begin{bmatrix} 1 & 3 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

SOLUTION:

$\triangleleft$

Exercise: Show that the determinant of upper or lower triangular matrices is the product of the diagonal coefficients.

Eigenvalues and Eigenvectors of a Matrix.

> **Definition 8.3.1.** A number λ and a nonzero n-vector $\boldsymbol{v}$ are an ***eigenvalue*** and ***eigenvector*** (eigenpair) of a square matrix A iff they satisfy the equation
>
> ________________________________.

Remarks:

(a) An eigenvector $\boldsymbol{v}$ determines a particular ______________________ in the space that remains ______________________ under the action of the matrix A.

(b) That is, if $\boldsymbol{v}$ is an eigenvector, so is __________________________________.

____.

Example 8.3.1: Verify that the pair λ_1, $\boldsymbol{v}_1$ and the pair λ_2, $\boldsymbol{v}_2$ are eigenvalue and eigenvector pairs of matrix A given below,

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \qquad \begin{cases} \lambda_1 = 4 & \boldsymbol{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \lambda_2 = -2 & \boldsymbol{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \end{cases}$$

Solution:

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Remark: How do we find the eigenvalues and eigenvectors of a square matrix?

Theorem 8.3.2. (Eigenvalues-Eigenvectors)

(a) All the eigenvalues λ of an $n \times n$ matrix A are the solutions of

____.

(b) Given an eigenvalue λ of an $n \times n$ matrix A, the corresponding eigenvectors $\boldsymbol{v}$ are the nonzero solutions to the homogeneous linear system

____.

Remark: An eigenvalue λ is a number such that $A - \lambda I$ is ________________________.

EXAMPLE 8.3.4: Find the eigenvalues λ and eigenvectors $\boldsymbol{v}$ of the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$.

SOLUTION:

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Diagonalizable Matrices.

Remark: We start with a preliminar concept: diagonal matrices.

Definition 8.3.5. An $n \times n$ matrix A is ***diagonal*** iff

.

____

Notation:

.

____

Theorem 8.3.6. If $D =$ ____________________________, then eigenpairs of D are

$, \cdots ,$.

______________________________ ________________________________

Remark: Now we can introduce diagonalizable matrices.

Definition 8.3.7. An square matrix A is ***diagonalizable*** iff there exists an invertible matrix P and a diagonal matrix D such that

_______________________________.

Remarks:

(a) A is diagonalizable iff __________________________, diagonal.

(b) e^A is ___________________________________ for A diagonalizable.

EXAMPLE 8.3.10: Show that matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ is diagonalizable, with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Find D.

SOLUTION:

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Remarks:

(1) $A = PDP^{-1}$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and ______________________________.

(2) Notice that ______________________ and ________________________, where

$$\boldsymbol{v}^{(1)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \boldsymbol{v}^{(2)} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \lambda_1 = 4, \quad \lambda_2 = -2,$$

____.

Theorem 3.8.8. (Diagonalizable Matrix) An $n \times n$ matrix A is diagonalizable iff A has n eigenvectors linearly independent. If $\lambda_i, \boldsymbol{v}_i$, for $i = 1, \cdots, n$, are eigenpairs of A, then $A = PDP^{-1}$, where

____________________, __________________________.

Example 8.3.11: Show that $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$, is diagonalizable.

Solution:

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Remark: Matrix P is ________________, since the eigenvectors are ________________. Another choice is

__________________________, ______________________________,

Show that ____________________ with

__________________ __________________.

The Exponential of a Matrix.

Definition 8.4.1. The ***exponential*** of a square matrix A is the infinite sum

$$\rule{5cm}{0.4pt}.$$

Remark: It can be shown that the infinite sum above converges for all square matrices.

EXAMPLE 8.4.1: Compute e^A, where $A = \begin{bmatrix} 2 & 0 \\ 0 & 7 \end{bmatrix}$.

SOLUTION:

$\triangleleft$

Theorem 8.4.3. If $D = \text{diag}\left[d_1, \cdots, d_n\right]$, then

____.

Remark: The exponential of a ______________________ matrix is simple to compute.

Theorem 8.4.5. If a square matrix A is diagonalizable, with $A = PDP^{-1}$ and D diagonal, then

________________________________.

Remark: To compute the exponential of a diagonalizable matrix we need to compute the _____________________ of that matrix.

Proof of Theorem 8.4.5:

□

EXAMPLE 8.4.2: Compute e^{At}, where $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ and $t \in \mathbb{R}$.

SOLUTION:

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Remark: Check that e^{At} above has the following property:

$$\underline{\hspace{10cm}}.$$