7.3. The Heat Equation

Section Objective(s):

- The Heat Equation (One-Space Dim).
- The IBVP: Dirichlet Conditions.
- The IBVP: Neumann Conditions.

7.3.1. The Heat Equation in (One-Space Dim).

**Definition 7.3.1.** The _______ equation in _______ dimension, for the function $u$ depending on _______ is

$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$, for _______, _______,

where $k > 0$ is a constant.

Remarks:

- $u$ is the _______ of a solid material.
- $t$ is _______, $x$ is _______.
- $k > 0$ is the _______, with units _______.
- The partial differential equation above has _______ solutions.
- We look for solutions satisfying both:
  - _______ conditions.
  - _______ conditions.

Boundary Conditions:  \hspace{2cm} Initial Conditions:
Remark: Qualitative behavior of solutions to the heat equation

\[ \text{...........................................} \]

Remarks: Generalizations and similar equations:

- The ______ equation in three space dimensions,

\[ \text{...........................................} \]

- The ______ equation in three space dimensions,

\[ \text{...........................................} \]

- The ________ equation of Quantum Mechanics,

\[ \text{...........................................} \]
7.3.2. The IBVP: Dirichlet Conditions.

**Theorem 7.3.2.** The BVP for the one-space dimensional heat equation,

\[
\text{---------------------, BC: ---------------------, ---------------------,}
\]

where \( k > 0, L > 0 \) are constants, has \___________\ many solutions

\[ \text{---------------------} \].

Furthermore, for every continuous function \( f \) on \([0, L]\) satisfying

\[ \text{---------------------} \], there is a unique solution \( u \) of the boundary value

problem above that also satisfies the \_________\ condition

\[ \text{---------------------} \].

This solution \( u \) is given by the expression above, where the coefficients \_____\ are

\[ \text{---------------------} \].

**Remarks:**

(a) This is an \___________\ Value Problem (IBVP).

(b) The boundary conditions are called \___________\ boundary conditions.

**Remark:** The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the \___________\ at the sides of the bar

is \_________.

(2) The initial condition is the \___________\ on the whole bar.

**Remark:** The proof is based on the \___________\.

(1) Look for \_________\ solutions of the \_________

(2) Any \___________\ of \_________\ solutions is also

\_________. (Superposition.)

(3) Determine the free constants with the \___________.
Proof of the Theorem:
**Example 7.3.1:** Find the solution to the initial-boundary value problem

\[ 4 \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 2], \]

with initial and boundary conditions given by

\[
\text{IC: } u(0, x) = \begin{cases} 
0 & x \in \left[0, \frac{2}{3}\right), \\
5 & x \in \left[\frac{2}{3}, \frac{4}{3}\right], \\
0 & x \in \left(\frac{4}{3}, 2\right], 
\end{cases} \quad \text{BC: } \begin{cases} 
0 & t = 0, \\
0 & t = 2.
\end{cases}
\]

**Solution:**
$\sin f(x) = f(x)$ for $x \in [0,2]$

So, therefore, solution of the initial-boundary-value problem for the equation $\sin (t,x) = 10\pi \sum_{n=1}^{\infty} \cos \frac{\pi}{3} \# - \cos \frac{2n\pi}{3} \# e^{-(n\pi/4)^2} \sin \frac{n\pi}{2} x \#$. \(\triangleright\)
7.3.3. The IBVP: Neumann Conditions.

**Theorem 7.3.3.** The BVP for the one-space dimensional heat equation,

\[
\begin{align*}
\text{equation here}, & \\
\text{BC: } & \text{condition here}, \text{ condition here}, \text{ condition here},
\end{align*}
\]

where \( k > 0, \) \( L > 0 \) are constants, has \( \text{number of solutions} \) many solutions

Furthermore, for every continuous function \( f \) on \([0, L]\) satisfying
\[
\text{condition here},
\]
there is a unique solution \( u \) of the boundary value problem above that also satisfies the \( \text{condition here} \) condition

\[
\text{condition here}.
\]

This solution \( u \) is given by the expression above, where the coefficients \( \text{number} \) are

\[
\text{expression here}.
\]

**Remarks:**

(a) This is an \( \text{type of problem} \) Value Problem (IBVP).

(b) The boundary conditions are called \( \text{type of condition} \) boundary conditions.

**Remark:** The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the \( \text{condition here} \) at the sides of the bar is \( \text{condition here} \).

(2) The initial condition is the \( \text{expression here} \) on the whole bar.

**Remark:** One can use \( \text{condition here} \) conditions on one side and \( \text{condition here} \) on the other side. This is called a \( \text{type of condition} \) boundary condition.

**Remark:** The proof is based on the \( \text{expression here} \).
Proof of the Theorem:
**Example 7.3.2:** Find the solution to the initial-boundary value problem

\[ \partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0, 3], \]

with initial and boundary conditions given by

\[
\text{IC: } u(0, x) = \begin{cases} 
7 & x \in \left[ \frac{3}{2}, 3 \right], \\
0 & x \in \left[ 0, \frac{3}{2} \right),
\end{cases} \quad \text{BC: } \begin{cases} 
u'(t, 0) = 0, \\
u'(t, 3) = 0.
\end{cases}
\]

**Solution:**