7.3. The Heat Equation

Section Objective(s):

- The Heat Equation (One-Space Dim).
- The IBVP: Dirichlet Conditions.
- The IBVP: Neumann Conditions.

7.3.1. The Heat Equation in (One-Space Dim).

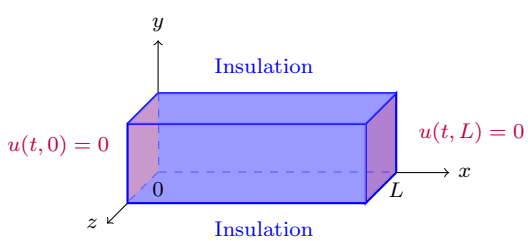
Definition 7.3.1. The _____ equation in _____ dimension, for the function u depending on _____ is

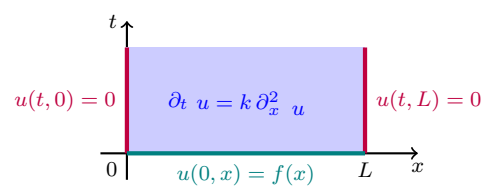
_____, for _____, _____,

where $k > 0$ is a constant.

Remarks:

- u is the _____ of a solid material.
- t is _____, x is _____.
- $k > 0$ is the _____, with units _____.
- The partial differential equation above has _____ solutions.
- We look for solutions satisfying both:
  - _____ conditions.
  - _____ conditions.



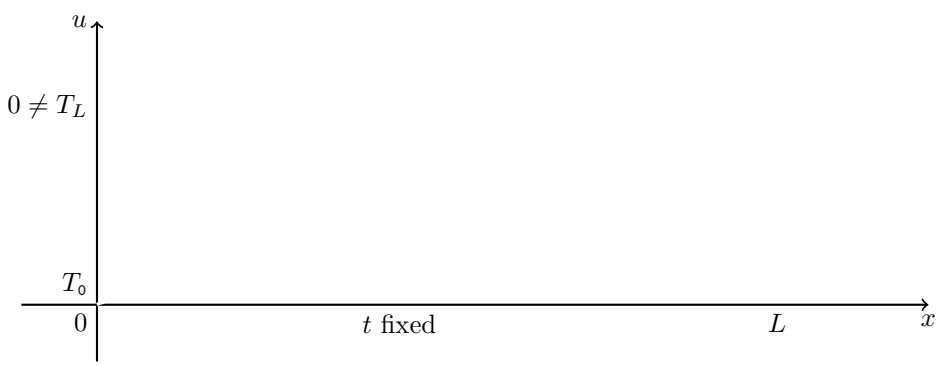


Boundary Conditions:

Initial Conditions:

Remark: Qualitative behavior of solutions to the heat equation

_____.



Remarks: Generalizations and similar equations:

- The _____ equation in three space dimensions,

  _____.

- The _____ equation in three space dimensions,

  _____.

- The _____ equation of Quantum Mechanics,

  _____.

7.3.2. **The IBVP: Dirichlet Conditions.**

Theorem 7.3.2. The BVP for the one-space dimensional heat equation,

_____, BC: _____, _____,

where $k > 0$, $L > 0$ are constants, has _____ many solutions

_____.

Furthermore, for every continuous function f on $[0, L]$ satisfying

_____, there is a unique solution u of the boundary value problem above that also satisfies the _____ condition

_____.

This solution u is given by the expression above, where the coefficients ____ are

_____.

Remarks:

(a) This is an _____ Value Problem (IBVP).

(b) The boundary conditions are called _____ boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the _____ at the sides of the bar is _____.

(2) The initial condition is the _____ on the whole bar.

Remark: The proof is based on the _____.

(1) Look for _____ solutions of the _____.

(2) Any _____ of _____ solutions is also _____. (Superposition.)

(3) Determine the free constants with the _____.

Proof of the Theorem:

Example 7.3.1: Find the solution to the initial-boundary value problem

$$4\,\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,2],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 0 & x \in \big[0, \frac{2}{3}\big), \\ 5 & x \in \big[\frac{2}{3}, \frac{4}{3}\big], \\ 0 & x \in \big(\frac{4}{3}, 2\big], \end{cases} \qquad \text{BC:} \quad \begin{cases} u(t,0) = 0, \\ u(t,2) = 0. \end{cases}$$

Solution:

◁

7.3.3. **The IBVP: Neumann Conditions.**

Theorem 7.3.3. The BVP for the one-space dimensional heat equation,

_____, BC: _____, _____,

where $k > 0$, $L > 0$ are constants, has _____ many solutions

_____.

Furthermore, for every continuous function f on $[0, L]$ satisfying

_____, there is a unique solution u of the boundary value problem above that also satisfies the _____ condition

_____.

This solution u is given by the expression above, where the coefficients ____ are

_____.

Remarks:

(a) This is an _____ Value Problem (IBVP).

(b) The boundary conditions are called _____ boundary conditions.

Remark: The physical meaning of the initial-boundary conditions is simple.

(1) The boundary conditions is to keep the _____ at the sides of the bar is _____.

(2) The initial condition is the _____ on the whole bar.

Remark: One can use _____ conditions on one side and _____ on the other side. This is called a _____ boundary condition.

Remark: The proof is based on the _____.

Proof of the Theorem:

□

Example 7.3.2: Find the solution to the initial-boundary value problem

$$\partial_t u = \partial_x^2 u, \quad t > 0, \quad x \in [0,3],$$

with initial and boundary conditions given by

$$\text{IC:} \quad u(0,x) = \begin{cases} 7 & x \in \left[\frac{3}{2}, 3\right], \\ 0 & x \in \left[0, \frac{3}{2}\right), \end{cases} \qquad \text{BC:} \quad \begin{cases} u'(t,0) = 0, \\ u'(t,3) = 0. \end{cases}$$

Solution:

$\triangleleft$