7.2. Overview of Fourier series

Section Objective(s):

- Fourier Expansion of Vectors.
- Fourier Expansion of Functions.
- Odd or Even Functions.
- Sine and Cosine Series.

7.2.1. Fourier Expansion of Vectors.

Remark: We review basic concepts about vectors in $\mathbb{R}^3$.

(a) The ________________ product of two vectors.

(b) ________________ and ________________ set of vectors.

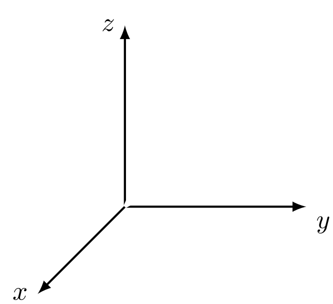
(c) Fourier ________________ (or orthonormal expansion) of vectors.

(d) Vector ____________________.

Definition 7.2.1. The ________________ of two vectors $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{R}^3$ is

________________________________,

with $|\boldsymbol{u}|$, $|\boldsymbol{v}|$ the magnitude of the vectors, and $\theta \in [0, \pi]$ the angle in between them.



Remarks:

- A vector $\boldsymbol{u}$ is a unit vector iff
$$\boldsymbol{u} \cdot \boldsymbol{u} = 1.$$
- The magnitude of a vector $\boldsymbol{u}$ can be written as
$$|\boldsymbol{u}| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}.$$

Remark: The dot product above satisfies the following properties.

Theorem 7.2.2. For every $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in \mathbb{R}^3$ and every a, $b \in \mathbb{R}$ holds,

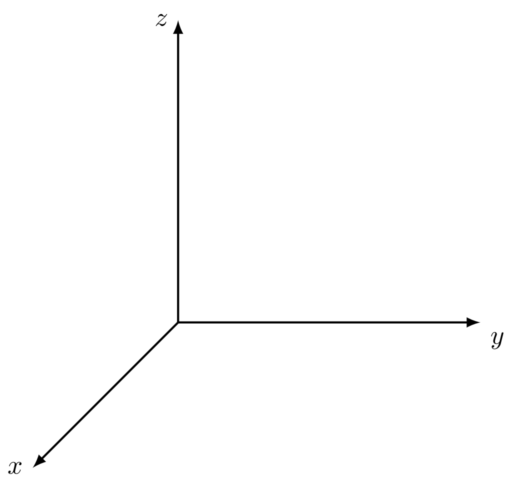
(a) $\boldsymbol{u} \cdot \boldsymbol{u} = 0$ iff $\boldsymbol{u} = 0$; and $\boldsymbol{u} \cdot \boldsymbol{u} > 0$ for $\boldsymbol{u} \neq 0$. ________________.

(b) $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$. ________________.

(c) $(a\boldsymbol{u} + b\boldsymbol{v}) \cdot \boldsymbol{w} = a\,(\boldsymbol{u} \cdot \boldsymbol{w}) + b\,(\boldsymbol{v} \cdot \boldsymbol{w})$. ________________.

Theorem 7.2.3. The vectors $\boldsymbol{u}$, $\boldsymbol{v}$ are ________________ iff $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

EXAMPLE 7.2.1: The set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an ______________________________ of $\mathbb{R}^3$.



Orthonormal means:

- Orthogonality.

- Normality.

$\triangleleft$

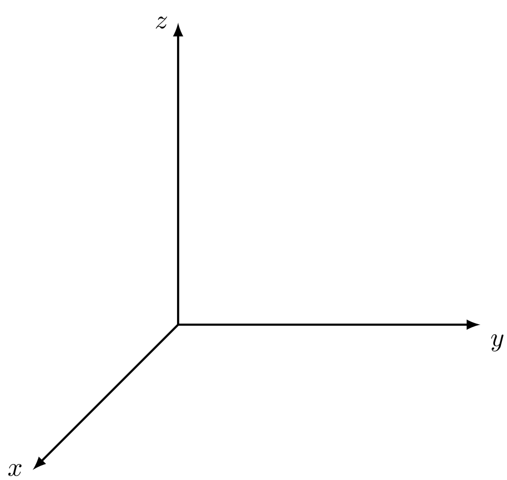
Theorem 7.2.4. (Fourier Expansion) The orthonormal set $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ is an orthonormal __________, that is, every $\boldsymbol{v} \in \mathbb{R}^3$ can be ____________________ as

____.

The orthonormality of the vector set implies a formula for the vector components

____________________, ____________________, ____________________.

Remark: The decomposition above allow us to introduce vector approximations.



Vector Approximations:

7.2.2. **Fourier Expansion of Functions.**

Remark: The ideas described above for vectors in $\mathbb{R}^3$ can be extended to functions.

Definition 7.2.5. The ______________ of two functions f, g on $[-L, L]$ is

______________________________.

Theorem 7.2.6. For every functions f, g, h and every a, $b \in \mathbb{R}$ holds,

(a) $f \cdot f = 0$ iff $f = 0$; and $f \cdot f > 0$ for $f \neq 0$. ____________.

(b) $f \cdot g = g \cdot f$. ____________.

(c) $(a\, f + b\, g) \cdot h = a\, (f \cdot h) + b\, (g \cdot h)$. ____________.

Remarks:

- The ____________ of a function f is

  ______________________________.

- A function f is a unit function iff ____________.

Definition 7.2.7. Two functions f, g are ______________ iff ______________.

Theorem 7.2.8. An ______________ in the space of continuous functions on $[-L, L]$ is

______________________________.

Remark: Orthogonality:

Remark: Normality:

Example 7.2.2: For $n \geqslant 1$ holds

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Remark: The orthogonality of the set above is a consequence of the following:

Theorem 7.2.9. (Orthogonality) The following relations hold for all n, $m \in \mathbb{N}$,

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \_\_ & n \neq m, \\ \_\_ & n = m \neq 0, \\ \_\_\_\_ & n = m = 0, \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \_\_ & n \neq m, \\ \_\_ & n = m, \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \_\_.$$

Proof:

Remark: Often in the literature is used the following ________________ set:

$$\underline{\hspace{10cm}}. \quad (7.2.1)$$

Theorem 7.2.10. (Fourier Expansion) The orthogonal set

$$\underline{\hspace{10cm}}.$$

is an orthogonal ________ of the space of ________________ functions on $[-L, L]$, that is, any continuous function on $[-L, L]$ can be __________________ as

$$\underline{\hspace{10cm}}.$$

Moreover, the coefficients above are given by the formulas

$$\underline{\hspace{9cm}},$$

$$\underline{\hspace{9cm}},$$

$$\underline{\hspace{9cm}}.$$

Furthermore, if f is ______________________________, then the function

$$\underline{\hspace{11cm}},$$

satisfies ____________________ for all x where f is ________________, while for all x_0 where f is ____________________ it holds

$$\underline{\hspace{10cm}}$$

EXAMPLE 7.2.3: Find the Fourier expansion of $f(x) = \begin{cases} \dfrac{x}{3}, & \text{for} \quad x \in [0,3] \\ 0, & \text{for} \quad x \in [-3,0). \end{cases}$

SOLUTION:

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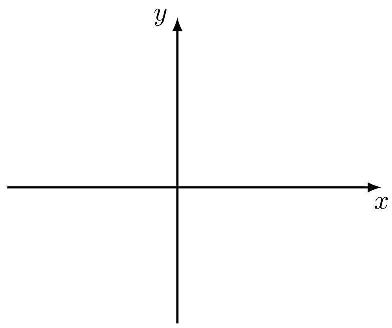
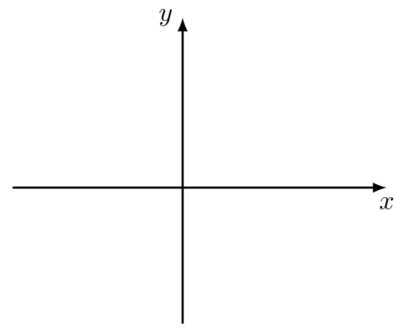
7.2.3. **Odd or Even Functions.**

Definition 7.2.11. A function f on $[-L, L]$ is:

- __________ iff ____________________ for all $x \in [-L, L]$;
- __________ iff ____________________ for all $x \in [-L, L]$.

EXAMPLE 7.2.4: The function $y = x^2$ is ________, while the function $y = x^3$ is ________.





Theorem 7.2.12. If f_e, g_e are even and h_o, ℓ_o are odd functions, then:

(1) $a\,f_e + b\,g_e$ is __________ for all a, $b \in \mathbb{R}$.

(2) $a\,h_o + b\,\ell_o$ is __________ for all a, $b \in \mathbb{R}$.

(3) $f_e\,g_e$ is __________.

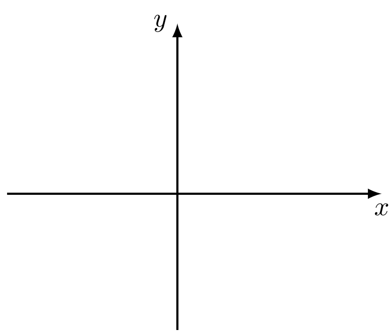
(4) $h_o\,\ell_o$ is __________.

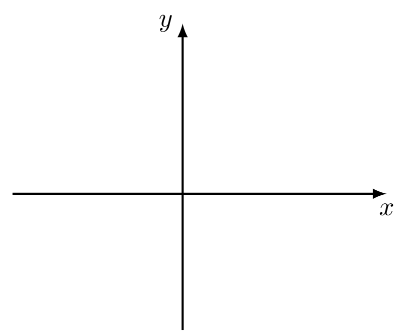
(5) $f_e\,h_o$ is __________.

(6) $\displaystyle\int_{-L}^{L} f_e\,dx =$ ____________________.

(7) $\displaystyle\int_{-L}^{L} h_o\,dx =$ ____________________.

Remark:





7.2.4. **Sine and Cosine Series.**

Theorem 7.2.13. Let f be a function on $[-L, L]$ with a Fourier expansion

.

____

(a) If f is __________, then ______________. The series

____

is called a _________________________.

(b) If f is ________, then ______________. The series

____

is called a _____________________.

Proof:

□

EXAMPLE 7.2.5: Find the Fourier expansion of $f(x) = \begin{cases} 1, & \text{for} \quad x \in [0,3] \\ -1, & \text{for} \quad x \in [-3,0). \end{cases}$

SOLUTION:

$\triangleleft$

EXAMPLE 7.2.6: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} x & x \in [0, 1], \\ -x & x \in [-1, 0). \end{cases}$$

SOLUTION:

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EXAMPLE 7.2.7: Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 1 - x & x \in [0, 1] \\ 1 + x & x \in [-1, 0). \end{cases}$$

SOLUTION:

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