7.1. Eigenfunction Problems

Section Objective(s):

- Two-Point Boundary Value Problems.
- Comparing IVP vs BVP.
- Eigenfunction Problems.

7.1.1. Two-Point Boundary Value Problems.

Definition. A ***two-point boundary value problem*** (BVP) is the following: Find solutions to the differential equation

$$y'' + a_1(x)\, y' + a_0(x)\, y = b(x)$$

satisfying the boundary conditions (BC)

____,

____,

where b_1, b_2, $\tilde{b}_1$, $\tilde{b}_2$, y_1, y_2, x_1, x_2 are given and $x_1 \neq x_2$.

Remarks:

(a) The two boundary conditions are held at different points, ____________.

(b) Both ______________ may appear in the boundary condition.

Example: We now show four examples of boundary value problems that differ only on the boundary conditions: Solve the different equation

$$y'' + a_1\, y' + a_0\, y = e^{-2t}$$

with the boundary conditions at $x_1 = 0$ and $x_2 = 1$ given below.

(a)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(b)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(c)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

(d)

$$\text{Boundary Condition:} \quad \begin{Bmatrix} y'(0) = y_1, \\ y'(1) = y_2, \end{Bmatrix} \quad \text{which is the case} \quad \begin{Bmatrix} b_1 = \_\_, & b_2 = \_\_, \\ \tilde{b}_1 = \_\_, & \tilde{b}_2 = \_\_. \end{Bmatrix}$$

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7.1.2. **Comparing IVP vs BVP.**

Definition 7.1.1. (IVP) Find a solution of $y'' + a_1\,y' + a_0\,y = 0$ satisfying the initial condition (IC)

$$\underline{\qquad\qquad\qquad\qquad}, \qquad \underline{\qquad\qquad\qquad\qquad}.$$

Remarks:

- The variable t represents ________.
- The variable y represents ______________.
- The IC are ______________ and ______________ at the initial time.

Definition 7.1.2. (BVP) Find a solution y of $y'' + a_1\,y' + a_0\,y = 0$ satisfying the boundary condition (BC)

$$\underline{\qquad\qquad\qquad\qquad}, \qquad \underline{\qquad\qquad\qquad\qquad}.$$

Remarks:

- The variable x represents ______________.
- The variable y may represent _____________________.
- The BC are _____________________ at two different _______________.

Theorem . The equation $y'' + a_1\,y' + a_0\,y = 0$ with IC $y(t_0) = y_0$ and $y'(t_0) = y_1$ has a ______________________________ for each choice of the IC.

Theorem 7.1.3. (BVP) The equation $y'' + a_1\,y' + a_0\,y = 0$ with BC $y(0) = y_0$ and $y(L) = y_1$, with $L \neq 0$ and with $r_\pm$ roots of $p(r) = r^2 + a_1 r + a_0$ satisfy the following:

(A) If $r_+ \neq r_-$, reals, then the BVP above has a __________________________.

(B) If $r_\pm$ are complex, then the solution of the BVP above belongs to only one of the following three possibilities:

  (i) There exists ____.

  (ii) There exists ____.

  (iii) There exists ____.

Proof of Theorem 7.1.3:

$\square$

EXAMPLE : Find all solutions to the BVPs $y'' + y = 0$ with the BCs:

$$\text{(a)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = 0. \end{cases} \qquad \text{(b)} \quad \begin{cases} y(0) = 1, \\ y(\pi/2) = 1. \end{cases} \qquad \text{(c)} \quad \begin{cases} y(0) = 1, \\ y(\pi) = -1. \end{cases}$$

SOLUTION:

$\triangleleft$

7.1.3. **Eigenfunction Problems.**

Remark: Let us recall the ***eigenvector*** problem of a square matrix: Given a square matrix A, find a number λ and a nonzero vector $\boldsymbol{v}$ solution of

____________________.

Definition . An ***eigenfunction problem*** is the following: Given a linear operator $L(y) = y'' + a_1\, y' + a_0\, y$, find a number λ and a nonzero function y solution of

____

and ______________________ boundary conditions at $x_1 \neq x_2$,

____,

____,

Remarks:

- Notice that _________ is always ________________ of the BVP above.
- Eigenfunctions are the ____________________________ of the BVP above.
- The eigenfunction problem is a BVP with ________________________ solutions.
- So, we look for ___ such that the operator _________________ has characteristic polynomial with _______________________.
- So, ___ is such that _________________ has _________________ solutions.
- We focus on the linear operator __________________________.

EXAMPLE : Find all numbers λ and nonzero functions y solutions of the BVP

$$y'' + \lambda\, y = 0, \qquad \text{with} \qquad y(0) = 0, \quad y(L) = 0, \qquad L > 0.$$

SOLUTION:

$\triangleleft$

EXAMPLE : Find the numbers λ and the nonzero functions y solutions of the BVP

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y'(L) = 0, \qquad L > 0.$$

SOLUTION:

EXAMPLE : Find the numbers λ and the nonzero functions y solutions of the BVP

$$x^2\, y'' - x\, y' = -\lambda\, y, \qquad y(1) = 0, \quad y(\ell) = 0, \quad \ell > 1.$$

SOLUTION: