5.3. 2×2 Constant coefficients systems

Section Objective(s):

- Diagonalizable systems.
  - Real Distinct Eigenvalues.
  - Complex Eigenvalues.
- Non-Diagonalizable systems.

5.3.1. Diagonalizable Systems.

Remark: We review the solutions of 2×2 diagonalizable systems.

Theorem 5.3.1. (Diagonalizable Systems) If the 2×2 constant matrix A is diagonalizable with eigenpairs $\lambda_{\pm}$, $\boldsymbol{v}^{(\pm)}$, then the general solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$ is

____.

Remark: We have three cases:

(i) The eigenvalues λ_+, λ_- are ______________________________.

(a) ____________________________,

(b) ____________________________,

(c) ____________________________.

(ii) The eigenvalues $\lambda_{\pm} = \alpha \pm \beta i$ ____;

(iii) The eigenvalues $\lambda_+ = \lambda_- = \lambda_0$ ____________________________________.

Remark: The case *(i)* was studied in the previous section.

EXAMPLE 5.3.1: Find the general solution of the 2×2 linear system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}.$$

SOLUTION:

◁

Remark: We now focus on case *(ii)*.

- A real matrix can have complex eigenvalues.
- But in this case, the eigenvalues—and the eigenvectors—come in pairs, $\lambda_+ = \overline{\lambda}_-$.

Theorem 5.3.2. (Conjugate Pairs) If A is a square matrix with real coefficients and λ, $\boldsymbol{v}$ is an eigenpair, then so is their complex conjugate $\overline{\lambda}$, $\overline{\boldsymbol{v}}$.

EXAMPLE: If a 2×2 matrix A has eigenpairs

$$\lambda = 7 + 2i, \quad \boldsymbol{v} = \begin{bmatrix} 2 + 3i \\ 5 - i \end{bmatrix},$$

then A also has the eigenpairs

.

◁

Proof of Theorem 5.3.2:

Theorem 5.3.3. (Complex and Real Solutions) If a 2×2 matrix A has eigenpairs

$$\lambda_{\pm} = \alpha \pm i\beta, \qquad \boldsymbol{v}^{(\pm)} = \boldsymbol{a} \pm i\boldsymbol{b},$$

where α, β, $\boldsymbol{a}$, and $\boldsymbol{b}$ real, then the equation $\boldsymbol{x}' = A\,\boldsymbol{x}$ has fundamental solutions

________________________________, ________________________________,

but it also has ***real-valued*** fundamental solutions

____,

____.

Proof of Theorem 5.3.3:

□

EXAMPLE 5.3.2: Find real-valued fundamental solutions to the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}.$$

SOLUTION:

◁

Remark: The case (iii), 2×2 diagonalizable matrices with a repeated eigenvalue.

Theorem 5.3.4. Every 2×2 diagonalizable matrix with repeated eigenvalue λ_0 has the form

__________________________.

Proof of Theorem 5.3.4:

$\square$

Remark: : The differential equation $\boldsymbol{x}' = \lambda_0 I\, \boldsymbol{x}$ is already ***decoupled***.

$$\left.\begin{aligned} x_1' &= \lambda_0\, x_1 \\ x_2' &= \lambda_0\, x_2 \end{aligned}\right\} \quad \Rightarrow \quad \text{too simple.}$$

5.3.2. **Non-Diagonalizable Systems.**

Remark: In this case is not so easy to find two fundamental solutions.

EXAMPLE : Find fundamental solutions to the system

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}$$

SOLUTION:

◁

Theorem 5.3.5. (Repeated Eigenvalue) If a 2×2 matrix A has a repeated eigenvalue λ with only one associated eigen-direction given by the eigenvector $\boldsymbol{v}$, then the differential system $\boldsymbol{x}'(t) = A\,\boldsymbol{x}(t)$ has a linearly independent set of solutions

________________________, ______________________________________,

where the vector _________ is one solutions of the algebraic linear system

_____________________________________.

Remark: For the proof see the Lecture Notes.

Example Similar to 5.3.3: Find the fundamental solutions of the differential equation

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad A = \begin{bmatrix} -6 & 4 \\ -1 & -2 \end{bmatrix}.$$

Solution:

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