5.2. Solution Formulas

Section Objective(s):

- Homogeneous Systems.
- Homogeneous Diagonalizable Systems.
- Non-Homogeneous Systems.

5.2.1. Homogeneous Systems.

Remark: The ______________________________ works for linear systems.

Theorem 5.2.1. (Homogeneous Systems) If A is an $n \times n$ constant matrix, then the initial value problem

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has a unique solution for all n-vectors $\boldsymbol{x}_0$ and all $t \in \mathbb{R}$, given by

______________________________.

Remark: Recall that e^{At}, for a constant square matrix A, satisfies:

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______________________________, ______________________________.

Proof of Theorem 5.2.1:

EXAMPLE 5.2.1: Compute the exponential function e^{At} and use it to express the vector-valued function $\boldsymbol{x}$ solution to the initial value problem

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0 = \begin{bmatrix} x_{01} \\ x_{02} \end{bmatrix}.$$

SOLUTION:

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5.2.2. **Homogeneous Diagonalizable Systems.**

Example Similar to 5.2.2: Find functions x_1, x_2 solutions of the first order, 2×2, constant coefficients, homogeneous differential system

$$
\begin{aligned}
x_1' &= x_1 + 3x_2, \\
x_2' &= 3x_1 + x_2.
\end{aligned}
$$

Solution:

$\triangleleft$

Theorem 5.2.2. (Homogeneous Diagonalizable Systems) If an $n \times n$ constant matrix A is diagonalizable, with linearly independent eigenvectors $\{\boldsymbol{v}^{(1)}, \cdots, \boldsymbol{v}^{(n)}\}$ and corresponding eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$, then the general solution of $\boldsymbol{x}' = A\,\boldsymbol{x}$ is

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Remark: Each function $\boldsymbol{x}^{(i)} = e^{\lambda_i t}\,\boldsymbol{v}^{(i)}$ is solution of the system $\boldsymbol{x}' = A\,\boldsymbol{x}$, because

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Proof of Theorem 5.2.2:

EXAMPLE SIMILAR TO 5.2.3: Use the theorem above to find the general solution of

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix}.$$

Then find $\boldsymbol{x}$ satisfying the initial condition $\boldsymbol{x}(0) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

SOLUTION:

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EXAMPLE : Follow the proof of the theorem above to find the general solution of

$$\boldsymbol{x}' = A\,\boldsymbol{x}, \qquad A = \begin{bmatrix} 3 & -2 \\ 10 & -6 \end{bmatrix}.$$

SOLUTION:

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Remark: The example above is similar to the homework problem below.

Consider the system $\mathbf{x}' = A\,\mathbf{x}$, with $A = \begin{bmatrix} 7 & -2 \\ 12 & -3 \end{bmatrix}$.

a. (1/10) Find the eigenvalues λ_+ and λ_-, larger and smaller or equal or conjugate, respectively, of the matrix A,

$\lambda_+ =$ [ ] $\lambda_- =$ [ ]

b. (1/10) Find corresponding eigenvectors $\mathbf{v}^+$ and $\mathbf{v}^-$, such that $\mathbf{v}^+$ has component $v_1^+ = 1$, and similarly for $\mathbf{v}^-$.

$\mathbf{v}^+ =$ [ ]

$\mathbf{v}^- =$ [ ]

c. (2/10) Compute the matrix $P = [\mathbf{v}^+, \mathbf{v}^-]$ and then introduce the new unknown function $\mathbf{y} = P^{-1}\mathbf{x}$, and denote its components as $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$.
Find the vector components y_1 and y_2 in terms of x_1 and x_2.

$y_1 =$ [ ] $x_1 +$ [ ] x_2

$y_2 =$ [ ] $x_1 +$ [ ] x_2

d. (2/10) Find the differential equations satisfied by the vector function $\mathbf{y}$, written in the form $\mathbf{y}' = B\,\mathbf{y}$.

$B = \begin{bmatrix} [\ ] & [\ ] \\ [\ ] & [\ ] \end{bmatrix}$

e. (2/10) Find the general solution $\mathbf{y}(t) = \begin{bmatrix} c_1\,\tilde{y}_1(t) \\ c_2\,\tilde{y}_2(t) \end{bmatrix}$ of the equation above, where c_1, c_2 are constants, and $\tilde{y}_1$, $\tilde{y}_2$ are fundamental solutions satisfying $\tilde{y}_1(0) = 1$ and $\tilde{y}_2(0) = 1$.

$\tilde{y}_1(t) =$ [ ]

$\tilde{y}_2(t) =$ [ ]

f. (2/10) Transform back to the unknown function $\mathbf{x} = P\,\mathbf{y}$. Write the solution as $\mathbf{x}(t) = c_1\,\mathbf{x}_1(t) + c_2\,\mathbf{x}_2(t)$, where c_1 and c_2 are the constants introduced above.

$\mathbf{x}_1(t) = \begin{bmatrix} [\ ] \\ [\ ] \end{bmatrix}$

$\mathbf{x}_2(t) = \begin{bmatrix} [\ ] \\ [\ ] \end{bmatrix}$

5.2.3. **Nonhomogeneous Systems.**

Theorem 5.2.5. (Nonhomogeneous Systems) If A is a constant $n \times n$ matrix and $\boldsymbol{b}$ is a continuous n-vector function, then the initial value problem

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has a unique solution for every initial condition $\boldsymbol{x}_0 \in \mathbb{R}^n$ given by

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Remark: The proof is based on the integrating factor method and follow the ideas of the proof for non-homogeneous scalar equations in Section 2.1.

Example 5.2.9: Find the vector-valued solution $\boldsymbol{x}$ to the differential system

$$\boldsymbol{x}' = A\,\boldsymbol{x} + \boldsymbol{b}, \quad \boldsymbol{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Solution:

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