4.4. Generalized Sources

Section Objective(s):

- The Dirac's Delta.
- Main Properties.
- Applications.
- The Impulse Response Function.

4.4.1. **The Dirac Delta.**

Definition 4.4.1. The ***Dirac delta*** generalized function is the limit

____,

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

____.

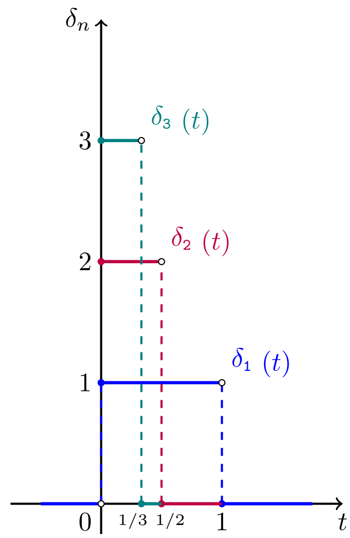
Remark: The sequence of bump functions introduced above can be rewritten as follows,

$$\delta_n(t) = \begin{cases} \underline{\qquad\qquad}, & t < 0 \\ \underline{\qquad\qquad}, & 0 \leqslant t < \dfrac{1}{n} \\ \underline{\qquad\qquad}, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} \underline{\qquad\qquad} & \text{for} \quad t \neq 0, \\ \underline{\qquad\qquad\quad} & \text{for} \quad t = 0. \end{cases}$$

Remark: There are infinitely many sequences $\{\delta_n\}$ of functions with the Dirac delta as their limit as $n \to \infty$.



Remarks:

(a) The Dirac delta is ______________________________ on the domain ______________.

(b) The Dirac delta is _________________________ on _________.

Theorem . Every function in the sequence $\{\delta_n\}$ above satisfies

____.

4.4.2. **Main Properties.**

Remark: We use ________________ to define operations on Dirac's deltas.

Definition 4.4.2. We introduce the following operations on the Dirac delta:

$$f(t)\,\delta(t-c) + g(t)\,\delta(t-c) = \underline{\hspace{10cm}}.$$

$$\int_a^b \delta(t-c)\,dt = \underline{\hspace{7cm}},$$

$$\mathcal{L}[\delta(t-c)] = \underline{\hspace{6cm}}.$$

Theorem 4.4.3. For every $c \in \mathbb{R}$ and $\epsilon > 0$ holds,

$$\underline{\hspace{9cm}}.$$

Proof of Theorem 4.4.3:

□

Theorem 4.4.4. If f is continuous on (a, b) and $c \in (a, b)$, then

____.

Proof of Theorem 4.4.4:

□

Theorem 4.4.5. For all $s \in \mathbb{R}$ holds

$$\mathcal{L}[\delta(t-c)] = \begin{cases} \underline{\hspace{5cm}} & \text{for} \quad c \geqslant 0, \\ \underline{\hspace{5cm}} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.4.5:

$\square$

4.4.3. **Applications of the Dirac Delta.**

Remarks:

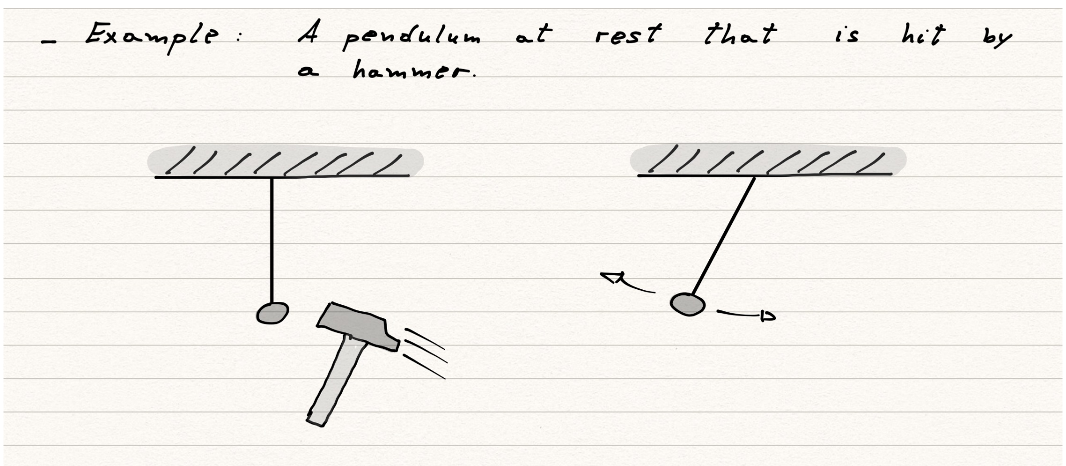
(a) Dirac's delta generalized function is useful to describe

____________________________________.

(b) An impulsive force transfers a ________________________________

in an ____.

(c) For example, a pendulum at rest that is hit by a hammer.



EXAMPLE 4.4.3: Use Newton's equation of motion and Dirac's delta to describe the change of momentum when a particle is hit by a hammer.

SOLUTION:

◁

4.4.4. **The Impulse Response Function.**

Definition 4.4.6. The ***impulse response function*** at the point $c \geqslant 0$ of the linear operator

______________________________,

with a_1, a_0 constants, is the solution y_δ of

______________, __________, __________.

Theorem 4.4.7. The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds

______________________.

where ______________________________________ of L.

Remark: The impulse response function y_δ at $c = 0$ satifies

______________________.

Proof of Theorem 4.4.7:

$\square$

EXAMPLE SIMILAR TO 4.4.6: Find the solution y to the initial value problem

$$y'' - y = \delta(t-3), \qquad y(0) = 0, \qquad y'(0) = 0.$$

SOLUTION:

◁