4.3. Discontinuous Sources

Section Objective(s):

- Review: Step Functions.
- Laplace Transform of Steps.
- Translation Properties of the LT.

4.3.1. **Step Functions.**

Definition 4.3.1. The ***step function*** at $t = 0$ is

$$u(t) = \begin{cases} \_\_\_\_\_\_ & t < 0, \\ \_\_\_\_\_\_ & t \geqslant 0. \end{cases}$$

Example 4.3.1: Graph the step u, $u_c(t) = u(t - c)$, and $u_{-c}(t) = u(t + c)$, for $c > 0$.

Solution:

◁

Example 4.3.2: Graph the bump function $b(t) = u(t - a) - u(t - b)$, for $a < b$.

Solution:

◁

4.3.2. **The Laplace Transform of Steps.**

Theorem 4.3.2. For every number $c \in \mathbb{R}$ and and every $s > 0$ holds

$$\mathcal{L}[u(t-c)] = \begin{cases} \underline{\hspace{4cm}} & \text{for} \quad c \geqslant 0, \\ \underline{\hspace{4cm}} & \text{for} \quad c < 0. \end{cases}$$

Proof of Theorem 4.3.2:

□

EXAMPLE 4.3.4: Compute $\mathcal{L}[3\,u(t-2)]$.

SOLUTION:

$\triangleleft$

Remarks:

(a) The LT is ______________________ transformation on the set of functions we work on.

(b) ____.

EXAMPLE 4.3.5: Compute $\mathcal{L}^{-1}\left[\dfrac{e^{-3s}}{s}\right]$.

SOLUTION:

$\triangleleft$

4.3.3. **Translation Identities.**

Theorem 4.3.3. (Translation Identities) If $\mathcal{L}[f(t)](s)$ exists for $s > a$, then

$$\mathcal{L}[u(t-c)f(t-c)] = \underline{\hspace{6cm}}, \qquad s > a, \qquad c \geqslant 0 \qquad (4.3.1)$$

$$\mathcal{L}[e^{ct}f(t)] = \underline{\hspace{6cm}}, \qquad s > a + c, \qquad c \in \mathbb{R}. \qquad (4.3.2)$$

EXAMPLE 4.3.6: Take $f(t) = \cos(t)$ and write the equations given the Theorem above.

SOLUTION:

$\triangleleft$

Remarks:

(a) We can highlight the main idea in the theorem above as follows:

$$\mathcal{L}\big[\text{right-translation } (uf)\big] = (\exp)\,\big(\mathcal{L}[f]\big),$$
$$\mathcal{L}\big[(\exp)\,(f)\big] = \text{translation}\big(\mathcal{L}[f]\big).$$

(b) Denoting $F(s) = \mathcal{L}[f(t)]$, then

____,

____.

(c) The inverse form of Eqs. (4.3.1)-(4.3.2) is

____,

____.

EXAMPLE 4.3.11: Find the function f such that $\mathcal{L}[f(t)] = \dfrac{e^{-4s}}{s^2+5}$.

SOLUTION:

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EXAMPLE 4.3.12: Find the function $f(t)$ such that $\mathcal{L}[f(t)] = \dfrac{(s-1)}{(s-2)^2+3}$.

SOLUTION:

◁

4.3.4. **Solving Differential Equations.**

Example 4.3.16: Use the LT to find the solution to the initial IVP

$$y'' + y' + \frac{5}{4}\,y = b(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad b(t) = \begin{cases} 1 & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.3}$$

Solution:

EXAMPLE 4.3.17: Use the LT to find the solution to the IVP

$$y'' + y' + \frac{5}{4}\,y = g(t), \qquad y(0) = 0, \qquad y'(0) = 0, \qquad g(t) = \begin{cases} \sin(t) & 0 \leqslant t < \pi \\ 0 & t \geqslant \pi. \end{cases} \tag{4.3.4}$$

SOLUTION:

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