4.1. Introduction to the Laplace Transform

Section Objective(s):

- Overview of the Method.
- The Laplace Transform.
- Main Properties.
- Solving a Differential Equation.

4.1.1. **Overview of the Method.**

Remark: The Laplace transform (LT) is a transformation: It changes a function into

____.

Example :

LT transforms	____________________	into	____________________.
Convention	variable $\to t$	into	variable $\to s$.
So LT transforms	____________________	into	____________________.

$\triangleleft$

Remark: Properties of the Laplace transform:

(a) The LT is a ____.

(b) The LT transforms ____________________ into __________________________.

Remark: The Laplace transform ($\mathcal{L}$) can be used to solve differential equations.

$$\mathcal{L}\left[\frac{\rule{3cm}{0.4pt}}{\text{eq. for } y(t).}\right] \xrightarrow{(1)} \frac{\rule{3cm}{0.4pt}}{\text{eq. for } \mathcal{L}[y(t)].} \xrightarrow{(2)} \begin{array}{c}\text{Solve the}\\ \rule{2.5cm}{0.4pt}\\ \text{eq. for } \mathcal{L}[y(t)].\end{array} \xrightarrow{(3)} \begin{array}{c}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Use the table.)}\end{array}$$

4.2.2. **The Laplace Transform.**

Definition 4.1.1. The ***Laplace transform*** of a function f on $D_f = [0, \infty)$ is

____,

defined for all $s \in D_F \subset \mathbb{R}$ where the ______________________________.

Remarks:

(a) Transformation notations for the Laplace transform: __________________.

____.

(b) Recall the definition of improper integrals:

$$\int_0^\infty g(t)\,dt = \underline{\hspace{6cm}}.$$

EXAMPLE 4.1.2: Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

SOLUTION:

$\lhd$

EXAMPLE 4.1.4: Compute $\mathcal{L}[\sin(at)]$, where $a \in \mathbb{R}$.

SOLUTION:

In Table 2 we present a short list of Laplace transforms. They can be computed in the same way we computed the the Laplace transforms in the examples above.

$f(t)$	$F(s) = \mathcal{L}[f(t)]$	D_F
$f(t) = 1$	$F(s) = \dfrac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \dfrac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \dfrac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \dfrac{a}{s^2 + a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \dfrac{s}{s^2 + a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \dfrac{a}{s^2 - a^2}$	$s > \|a\|$
$f(t) = \cosh(at)$	$F(s) = \dfrac{s}{s^2 - a^2}$	$s > \|a\|$
$f(t) = t^n e^{at}$	$F(s) = \dfrac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \dfrac{b}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 + b^2}$	$s > a$
$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2 - b^2}$	$s - a > \|b\|$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2 - b^2}$	$s - a > \|b\|$

TABLE 2. The Laplace transform of a few functions.

4.1.3. **Main Properties.**

Remark: We summarize three main properties of the Laplace transform.

Theorem 4.1.3. (Convergence of LT) If f on $[0, \infty)$ is piecewise continuous and bounded by an exponential, that is, there is k, $a > 0$ such that

$$\rule{6cm}{0.4pt},$$

then $\rule{8cm}{0.4pt}$.

Remark: An example of a function that is not bounded by an exponential is

$$\rule{4cm}{0.4pt}.$$

Theorem 4.1.4. (Linearity) If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all a, $b \in \mathbb{R}$ holds

$$\rule{9cm}{0.4pt}.$$

Theorem 4.1.5. (Derivative into Multiplication) If a function f is continuously differentiable on $[0, \infty)$ and $|f(t)| \leqslant k\, e^{at}$, then $\mathcal{L}[f']$ exists for $s > a$ and

$$\rule{9cm}{0.4pt}.$$

Exercise: Use the formula above to compute the LT of higher derivatives,

$$\mathcal{L}[f''] = \rule{9cm}{0.4pt}.$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = \rule{9cm}{0.4pt}.$$

Proof of Theorem 4.1.5:

□

4.1.4. **Solving a Differential Equation.**

Remark: The Laplace transform ($\mathcal{L}$) can be used to solve differential equations.

$$\mathcal{L}\left[\underset{\text{eq. for } y(t).}{\underline{\hspace{3cm}}}\right] \xrightarrow{(1)} \underset{\text{eq. for } \mathcal{L}[y(t)].}{\underline{\hspace{3cm}}} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \underline{\hspace{2.5cm}}\\ \text{eq. for } \mathcal{L}[y(t)].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y(t).\\ \text{(Use the table.)}\end{matrix}$$

EXAMPLE 4.1.8: Use the Laplace transform to find y solution of

$$y'' + 9\,y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

Remark: We know what the solution of this problem is.

SOLUTION: