## 2.5. Nonhomogeneous Equations

**Section Objective(s):**

- The General Solution Theorem.
- Computing a Particular Solution $y_p$.
  - Undetermined Coefficients.
  - Variation of Parameters.

### 2.5.1. **The General Solution Theorem.**

**Remarks:**

- The General Solution Theorem proven for homogeneous equations

$$L(y) = 0, \qquad \text{with} \qquad L(y) = y'' + a_1(t)\, y' + a_0(t)\, y,$$

  is ____________________ for nonhomogeneous equations $L(y) = f$, with $f \neq 0$.

- The superposition property is _______________ for nonhomogeneous equations.

- Recall the superposition property for $L(y) = 0$:
  - If $L(y_1) = 0$ and $L(y_2) = 0$ then

- But for nonhomogeneous equations $L(y) = f$:
  - If $L(y_1) = f$ and $L(y_2) = f$ then

**Theorem 2.5.1.** (General Solution) If $y_1$ and $y_2$ are fundamental solutions of

________________________________________

where __________________________________, and ________ is one solution of ______________, then all solutions of the ________________________ equation _____________ are

___________________________________________________.

**Definition 2.5.2.** The ***general solution*** of $L(y) = f$ is

__________________________________________________

where $y_1$, $y_2$ are fundamental solutions of $L(y) = 0$, and

___________________________.

**Proof of Theorem 2.5.1:**

$\square$

2.5.2. **The Undetermined Coefficients Method.**

**Problem:**
Find a function ________ solution of ____________________,
where $L(y) = y'' + a_1\, y' + a_0\, y$ and $a_1, a_0 \in \mathbb{R}$.

**Idea:**

If ________________, then try ________________, and find ___.

If ________________, then try ________________.

If ________________, then try ________________.

If ____________, then try ________________________________.

If ______________________,

then try ________________________________________________.

**Summary of the Undetermined Coefficients Method:**

(1) Find fundamental solutions $y_1$, $y_2$ of the homogeneous equation $L(y) = 0$.

(2) Given the source functions $f$, guess the solutions $y_p$ following the Table 1 below.

(3) If ________ given by the table satisfies ______________,
then change the guess to ________________.

(4) If ________ satisfies ___________________________________,
then change the guess to __________________.

(5) Find the constants $k$ in the function $y_p$ using the equation _____________.

| $f(t)$ (Source) ($K$, $m$, $a$, $b$, given.) | $y_p(t)$ (Guess) ($k$ not given.) |
|---|---|
| $Ke^{at}$ | $ke^{at}$ |
| $K_m t^m + \cdots + K_0$ | $k_m t^m + \cdots + k_0$ |
| $K_1 \cos(bt) + K_2 \sin(bt)$ | $k_1 \cos(bt) + k_2 \sin(bt)$ |

TABLE 1. List of sources $f$ and solutions $y_p$ to the equation $L(y_p) = f$.

EXAMPLE 2.5.1: (First Guess Right) Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{2t}.$$

SOLUTION:

◁

**Remark:** The step (4) in Example 2.5.1 is a particular case of the following statement.

**Theorem 2.5.3.** The constant coefficients nonhomogeneous equation

______________________________________________

with ____________________, where $p(r) = r^2 + a_1 r + a_0$, has the particular solution

______________________________________________

**Remark:** As we said, the step (4) in Example 2.5.1 is a particular case of Theorem 2.5.3,

**Proof of Theorem 2.5.3:**

□

EXAMPLE 2.5.2: (First Guess Wrong) Find all solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 3\,e^{4t}.$$

SOLUTION:

$\triangleleft$

EXAMPLE 2.5.3: (First Guess Right) Find all the solutions to the nonhomogeneous equation

$$y'' - 3y' - 4y = 2\sin(t).$$

SOLUTION:

◁

### 2.5.3. The Variation of Parameters Method.

**Remarks:**

- Variation of Parameters Method (VPM) works on ____________________ equations than the undetermined coefficients method (UCM).
- VPM works on ______________________________________
- VPM usually ____________________ to implement than the UCM.

**Theorem 2.5.4.** (Variation of Parameters) A particular solution to the equation

$$L(y) = f,$$

with $L(y) = y'' + a_1(t)\, y' + a_0(t)\, y$ and $a_1$, $a_0$, $f$ continuous functions, is given by

____________________________________,

where $y_1$, $y_2$ are fundamental solutions of $L(y) = 0$ and ____________ are

__________________________________________________,

where __________ is the __________________ of $y_1$ and $y_2$.

**Remark:** The proof is based on a generalization of the reduction of order method.

**Proof of Theorem 2.5.4:**

**Remark:** The integration constants in __________ can always be chosen ________.

EXAMPLE 2.5.6: Find the general solution of the nonhomogeneous equation

$$y'' - 5y' + 6y = 2\,e^{t}.$$

SOLUTION:

◁

EXAMPLE 2.5.7: Find a particular solution to the differential equation

$$t^2y'' - 2y = 3t^2 - 1,$$

knowing that $y_1 = t^2$ and $y_2 = 1/t$ are solutions to the homogeneous equation $t^2y'' - 2y = 0$.

SOLUTION:

◁