2.2. Reduction of Order Methods

Section Objective(s):

- Special Nonlinear Equations.
  - Function y Missing.
  - Variable t Missing.
- Conservation of the Energy.
  - Variable t and Function y' Missing.
- The Reduction of Order Method.

2.2.1. Special Nonlinear Equations.

Definition 2.2.1. A second order equation $y'' = f(t, y, y')$ is ***special*** iff one of the following equations hold,

____.

Theorem 2.2.2. (Function y Missing) If a second order differential equation has the form $y'' = f(t, y')$, then $v = y'$ satisfies the first order equation

______________________________.

Proof of Theorem 2.2.2: Left as exercise.

Example 2.2.1: Solve $y'' = -2t\,(y')^2$ with initial conditions $y(1) = 2$, $y'(1) = 1$.

Solution:

◁

Theorem 2.2.3. (Variable t Missing) If the initial value problem

$$\rule{9cm}{0.4pt},$$

has an invertible solution y then the function

$$\rule{9cm}{0.4pt},$$

where $v(t) = y'(t)$ and $t(y)$ is the inverse of $y(t)$, satisfies the initial value problem

$$\rule{9cm}{0.4pt}.$$

Proof of Theorem 2.2.3:

$\square$

Example 2.2.3: Find the solution y to the initial value problem

$$y\,y'' + 3(y')^2 = 0, \qquad y(0) = 1, \quad y'(0) = 6.$$

Solution:

2.2.2. **Conservation of the Energy.**

Remark: When a force f depends only on the particle position y, we have

$$\underline{\hspace{6cm}},$$

where m is the particle mass. So f is special, ____________________________.

In this case the mechanical energy is ____________________.

Theorem 2.2.4. (Conservation of the Energy) Consider a particle with positive ____________ and ____________________, function of _______________, which is a solution of Newton's law of motion

$$\underline{\hspace{8cm}},$$

with initial conditions ____,
where _______ is the force acting on the particle at the _____________________.
Then, the position function y satisfies

$$\underline{\hspace{8cm}},$$

where

$$\underline{\hspace{8cm}}$$

is fixed by the initial conditions, ______________________ is the particle velocity, and ___ is the potential of the force f—the negative of the primitive of f, in other words,

$$\underline{\hspace{11cm}}.$$

Remarks:

- __________________________ is the kinetic energy of the particle.
- __________________ is the potential energy.
- _______________________________ is the mechanical energy.

Proof of Theorem 2.2.4:

□

EXAMPLE 2.2.4: Find the potential energy and write the energy conservation for:

(i) A particle attached at $y = 0$ to a spring with constant k, moving in one space dimension on the y axis. In this case the force on the particle is $f(y) = -ky$.

(ii) A particle moving vertically on Earth's constant gravitational acceleration. In this case the force on the particle having mass m is $f(y) = mg$, where $g = 9.81\ \mathrm{m/s}^2$.

SOLUTION:

◁

EXAMPLE 2.2.5: Find the maximum height of a ball of mass $m = 0.1$ Kg that is shot vertically by a spring with spring constant $k = 400$ Kg/s^2 and compressed 0.1 m. Use $g = 10$ m/s^2.

SOLUTION:

◁

2.2.3. **The Reduction of Order Method.**

Theorem 2.2.5. (Reduction of Order) If a nonzero function y_1 is solution to

$$y'' + a_1\, y' + a_0\, y = 0. \tag{2.2.1}$$

where a_1, a_0 are given functions, then a second solution not proportional to y_1 is

____,

where ___________________________.

Remark: In the first part of the proof we write ___________________________ and show that y_2 is solution of Eq. (2.2.1) iff the function v is solution of

____. (2.2.2)

Proof of Theorem 2.2.5:

□

Example 2.2.8: Find a second solution y_2 linearly independent to the solution $y_1(t) = t$ of the differential equation

$$t^2\, y'' + 2t\, y' - 2y = 0.$$

Solution:

◁