Section Objective(s):

- Definitions and Examples.
- Solutions to the Initial Value Problem.
- Properties of Homogeneous Equations.
- The General Solution Theorem.
- The Wronskian and Abel's Theorem.

### 2.1.1. Definitions and Examples.

**Definition 2.1.1.** A *second order linear* differential equation on *y* is

where  $a_1, a_0, b$  are given functions. The differential equation above:

- (a) is *homogeneous* iff the source \_\_\_\_\_\_ for all  $t \in \mathbb{R}$ ;
- (b) has *constant coefficients* iff are constants;
- (c) has *variable coefficients* iff either \_\_\_\_\_\_ is not constant.

**Remark:** The homogeneous equations here \_\_\_\_\_\_ the Euler homogeneous equations in § 1.3.

#### Example 2.1.1:

- (a) A second order, linear, homogeneous, constant coefficients equation is
- (b) A second order, linear, nonhomogeneous, constant coefficients, equation is
- (c) A second order, linear, nonhomogeneous, variable coefficients equation is
- (d) Newton's law of motion for a point particle of mass m moving in one space dimension under a force f is mass times acceleration equal force,
- (e) Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

where \_\_\_\_\_ is the probability density of finding a particle of mass \_\_\_\_\_\_ at the position \_\_\_\_\_\_ having energy \_\_\_\_\_\_ under a potential \_\_\_\_\_\_, where  $\hbar$  is the Planck constant divided by  $2\pi$ .  $\triangleleft$ 

EXAMPLE 2.1.3: Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2 t, \qquad c_1, c_2 \in \mathbb{R}.$$

SOLUTION:

**Proof:** Based on the Picard iteration. Two integrations, two initial conditions.

EXAMPLE 2.1.5: Find the domain of the solution to the initial value problem

$$(t-1)y'' - 3ty' + \frac{4(t-1)}{(t-3)}y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

SOLUTION:

## 2.1.3. Properties of Homogeneous Equations.

**Remark:** We introduce a new notation. We write

 $y'' + a_1(t) y' + a_0(t) y = b(t)$  as

where

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Here \_\_\_\_\_ is an operator, that is,

The operator above is a linear operator.

**Definition 2.1.3.** A *linear operator* is an operator L such that for every pair of functions  $y_1$ ,  $y_2$  and constants  $c_1$ ,  $c_2$  holds

EXAMPLE SIMILAR TO THEOREM 2.1.4: Show that the operator  $L(y) = y'' + a_1 y' + a_0 y$  is a linear operator.

SOLUTION:

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## Remark:

Linearity of an operator and the superposition property are two sides of the same coin.

**Theorem 2.1.5.** (Superposition Property) If L is a linear operator and  $y_1$ ,  $y_2$  are solutions of the homogeneous equations

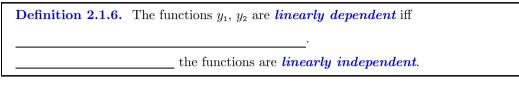
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then for every constants  $c_1,\,c_2$  holds

**Remark:** This result \_\_\_\_\_\_ for nonhomogeneous equations.

Proof of Theorem 2.1.5:

## 2.1.4. The General Solution Theorem.



**Remark:** Two functions  $y_1, y_2$  are proportional iff there is a constant c such that

**Theorem 2.1.7.** (General Solution) If  $y_1$  and  $y_2$  are linearly independent solutions of the variable coefficients equation

$$L(y) = y'' + a_1 y' + a_0 y,$$

then solution y of L(y) = 0 can be written as

Definition 2.1.8.

- (a) The functions  $y_1$  and  $y_2$  are *fundamental solutions* of L(y) = 0 iff holds that  $L(y_1) = 0, L(y_2) = 0$  and  $y_1, y_2$  are .
- (b) The *general solution* of the homogeneous equation L(y) = 0 is a two-parameter family of functions

where  $y_1, y_2$  are \_\_\_\_\_

f L(y) = 0.

EXAMPLE 2.1.8: Show that  $y_1 = e^t$  and  $y_2 = e^{-2t}$  are fundamental solutions to the equation y'' + y' - 2y = 0.

SOLUTION:

**Remark:** The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

Proof of Theorem 2.1.7:

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**Definition 2.1.9.** The *Wronskian* of functions  $y_1, y_2$  is the function

**Remark:** If  $A(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y'_1(t) & y'_2(t) \end{bmatrix}$ , then \_\_\_\_\_\_.

EXAMPLE SIMILAR TO 2.1.9: Find the Wronskian of  $y_1 = e^{2t}$  and  $y_2 = e^{3t}$ .

SOLUTION:

**Theorem 2.1.10.** (Wronskian I) If  $y_1, y_2$  are linearly dependent, then \_\_\_\_\_

Proof of Theorem 2.1.10:

# **Remark:**

If  $y_1$  and  $y_2$  are linearly independent, then it \_\_\_\_\_ imply that  $W_{12} \neq 0$ .

EXAMPLE 2.1.10: Show that the functions  $y_1(t) = t^2$  and  $y_2(t) = |t| t$ , for  $t \in \mathbb{R}$ , are linearly independent and have Wronskian  $W_{12} = 0$ .

SOLUTION:

**Theorem 2.1.12.** (Abel) If  $y_1, y_2$  are twice continuously differentiable solutions of  $y'' + a_1(t) y' + a_0(t) y = 0,$ 

(2.1.1)

 $\triangleleft$ 

where  $a_1, a_0$  are continuous on  $I \subset \mathbb{R}$ , then the Wronskian  $W_{12}$  satisfies

Therefore, for any  $t_0 \in I$ , the Wronskian  $W_{12}$  is given by the expression

where  $W_0 = W_{12}(t_0)$  and

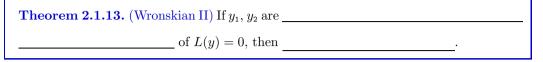
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Proof of Theorem 2.1.12:

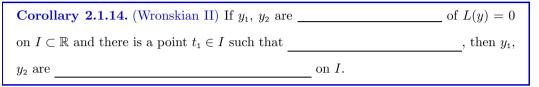
EXAMPLE 2.1.11: Find the Wronskian of two solutions of the equation  $t^2\,y''-t(t+2)\,y'+(t+2)\,y=0,\qquad t>0.$ 

SOLUTION:

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**Remark:** Instead of proving the Theorem above, we prove an equivalent statement—the negative statement.



Proof of Corollary 2.1.14: