2.1. Variable Coefficients

Section Objective(s):

- Definitions and Examples.
- Solutions to the Initial Value Problem.
- Properties of Homogeneous Equations.
- The General Solution Theorem.
- The Wronskian and Abel's Theorem.

2.1.1. Definitions and Examples.

Definition 2.1.1. A ***second order linear*** differential equation on y is

$$\rule{8cm}{0.4pt},$$

where a_1, a_0, b are given functions. The differential equation above:

(a) is ***homogeneous*** iff the source __________________ for all $t \in \mathbb{R}$;

(b) has ***constant coefficients*** iff ____________________ are constants;

(c) has ***variable coefficients*** iff either __________________ is not constant.

Remark: The homogeneous equations here ____________________________________ the Euler homogeneous equations in § 1.3.

Example 2.1.1:

(a) A second order, linear, homogeneous, constant coefficients equation is

$$\rule{8cm}{0.4pt}.$$

(b) A second order, linear, nonhomogeneous, constant coefficients, equation is

$$\rule{8cm}{0.4pt}.$$

(c) A second order, linear, nonhomogeneous, variable coefficients equation is

$$\rule{8cm}{0.4pt}.$$

(d) Newton's law of motion for a point particle of mass m moving in one space dimension under a force f is mass times acceleration equal force,

$$\rule{8cm}{0.4pt}.$$

(e) Schrödinger equation in Quantum Mechanics, in one space dimension, stationary, is

$$\rule{8cm}{0.4pt},$$

where ___ is the probability density of finding a particle of mass ____ at the position ___ having energy ____ under a potential ____, where $\hbar$ is the Planck constant divided by 2π. $\triangleleft$

EXAMPLE 2.1.3: Find the differential equation satisfied by the family of functions

$$y(t) = \frac{c_1}{t} + c_2\, t, \qquad c_1, c_2 \in \mathbb{R}.$$

SOLUTION:

$\triangleleft$

2.1.2. **Solutions to the Initial Value Problem.**

Theorem 2.1.2. (IVP) If the a_1, a_0, b are continuous on (t_1, t_2) and $t_0 \in (t_1, t_2)$, then there is ________________________________ solution of the initial value problem

____.

Proof: Based on the Picard iteration. Two integrations, two initial conditions.

EXAMPLE 2.1.5: Find the domain of the solution to the initial value problem

$$(t-1)\,y'' - 3t\,y' + \frac{4(t-1)}{(t-3)}\,y = t(t-1), \qquad y(2) = 1, \qquad y'(2) = 0.$$

SOLUTION:

$\triangleleft$

2.1.3. **Properties of Homogeneous Equations.**

Remark: We introduce a new notation. We write

$$y'' + a_1(t)\, y' + a_0(t)\, y = b(t) \qquad \text{as} \qquad \underline{\hspace{6cm}},$$

where

$$\underline{\hspace{12cm}}.$$

Here ____ is an operator, that is,

$$\underline{\hspace{12cm}}.$$

The operator above is a linear operator.

Definition 2.1.3. A ***linear operator*** is an operator L such that for every pair of functions y_1, y_2 and constants c_1, c_2 holds

$$\underline{\hspace{14cm}}.$$

Example Similar to Theorem 2.1.4: Show that the operator $L(y) = y'' + a_1\, y' + a_0\, y$ is a linear operator.

Solution:

$\triangleleft$

Remark:
Linearity of an operator and the superposition property are two sides of the same coin.

Theorem 2.1.5. (Superposition Property) If L is a linear operator and y_1, y_2 are solutions of the homogeneous equations

$$\underline{\hspace{6cm}},$$

then for every constants c_1, c_2 holds

$$\underline{\hspace{6cm}}.$$

Remark: This result ____________________ for nonhomogeneous equations.

Proof of Theorem 2.1.5:

$\square$

2.1.4. **The General Solution Theorem.**

Definition 2.1.6. The functions y_1, y_2 are ***linearly dependent*** iff

____.

______________________ the functions are ***linearly independent***.

Remark: Two functions y_1, y_2 are proportional iff there is a constant c such that

____.

Theorem 2.1.7. (General Solution) If y_1 and y_2 are linearly independent solutions of the variable coefficients equation

____, $\qquad L(y) = y'' + a_1\, y' + a_0\, y,$

then ______________ solution y of $L(y) = 0$ can be written as

____.

Definition 2.1.8.

(a) The functions y_1 and y_2 are ***fundamental solutions*** of $L(y) = 0$ iff holds that $L(y_1) = 0$, $L(y_2) = 0$ and y_1, y_2 are _________________________________.

(b) The ***general solution*** of the homogeneous equation $L(y) = 0$ is a two-parameter family of functions

____,

where y_1, y_2 are ____ of $L(y) = 0$.

Example 2.1.8: Show that $y_1 = e^t$ and $y_2 = e^{-2t}$ are fundamental solutions to the equation

$$y'' + y' - 2y = 0.$$

Solution:

$\lhd$

Remark: The fundamental solutions to the equation above are not unique. For example, show that another set of fundamental solutions to the equation above is given by,

.

Proof of Theorem 2.1.7:

□

2.1.5. **The Wronskian and Abel's Theorem.**

Definition 2.1.9. The ***Wronskian*** of functions y_1, y_2 is the function

____.

Remark: If $A(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$, then _________________________________.

Example Similar to 2.1.9: Find the Wronskian of $y_1 = e^{2t}$ and $y_2 = e^{3t}$.

Solution:

$\triangleleft$

Theorem 2.1.10. (Wronskian I) If y_1, y_2 are linearly dependent, then _____________.

Proof of Theorem 2.1.10:

$\square$

Remark:
If y_1 and y_2 are linearly independent, then it __________________ imply that $W_{12} \neq 0$.

EXAMPLE 2.1.10: Show that the functions $y_1(t) = t^2$ and $y_2(t) = |t|\, t$, for $t \in \mathbb{R}$, are linearly independent and have Wronskian $W_{12} = 0$.

SOLUTION:

$\triangleleft$

Theorem 2.1.12. (Abel) If y_1, y_2 are twice continuously differentiable solutions of

$$y'' + a_1(t)\, y' + a_0(t)\, y = 0, \tag{2.1.1}$$

where a_1, a_0 are continuous on $I \subset \mathbb{R}$, then the Wronskian W_{12} satisfies

____.

Therefore, for any $t_0 \in I$, the Wronskian W_{12} is given by the expression

____,

where $W_0 = W_{12}(t_0)$ and ____.

Proof of Theorem 2.1.12:

$\square$

EXAMPLE 2.1.11: Find the Wronskian of two solutions of the equation

$$t^2\, y'' - t(t+2)\, y' + (t+2)\, y = 0, \qquad t > 0.$$

SOLUTION:

◁

Theorem 2.1.13. (Wronskian II) If y_1, y_2 are ____ __________________________ of $L(y) = 0$, then ______________________________.

Remark: Instead of proving the Theorem above, we prove an equivalent statement—the negative statement.

Corollary 2.1.14. (Wronskian II) If y_1, y_2 are _________________________ of $L(y) = 0$ on $I \subset \mathbb{R}$ and there is a point $t_1 \in I$ such that _____________________________, then y_1, y_2 are ____ on I.

Proof of Corollary 2.1.14: