1.5. Applications

Section Objective(s):

- The Radioactive Decay Equation.
- Newton's Cooling Law.
- Salt in a Water Tanks.

1.5.1. Exponential Decay.

Definition 1.5.1. The ***exponential decay*** equation for N is

______________________________________.

Remark: The *exponential growth* equation is ______________________________________.

Theorem 1.5.1. The solution of the exponential decay equation with $N(0) = N_0$ is

__________________________.

Proof of Theorem 1.5.1:

This is a(n) ____________ equation with ____.

$\square$

Remark: Radioactive materials are often characterized by their ___________________.

Definition 1.5.2. The ________________ of a radioactive material with an initial amount N_0 is the time τ such that

____________________.

Theorem 1.5.2. A radioactive material constant k and half-life τ are related by

__________________.

Proof of Theorem 1.5.2:

$\square$

EXAMPLE 1.5.1: If certain remains are found containing an amount of 14 % of the original amount of Carbon-14, find the date of the remains.

SOLUTION:

◁

1.5.2. **Newton's Cooling Law.**

Definition 1.5.3. The ***Newton cooling law*** says that the temperature T at a time t of a material placed in a medium with constant temperature T_s satisfies

____

where $\Delta T(t) = T(t) - T_s$, and $k > 0$, constant.

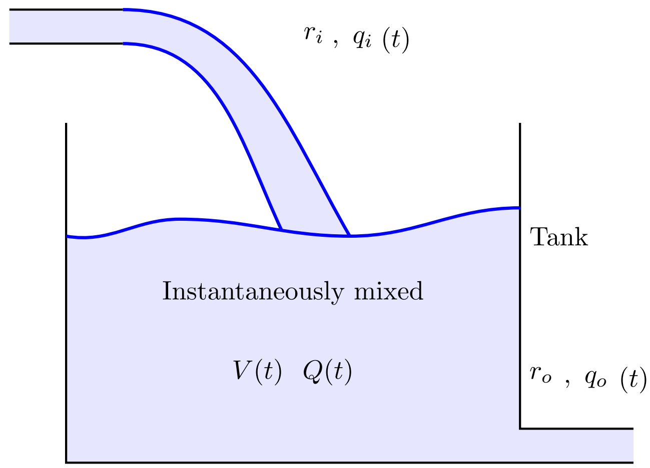
Remark:

EXAMPLE 1.5.2: A cup with water at 45 C is placed in the cooler held at 5 C. If after 2 minutes the water temperature is 25 C, when will the water temperature be 15 C?

SOLUTION:

◁

1.5.3. **Salt in a Water Tank.**



Remark: Before stating the problem we want to solve, we review the physical units of the main fields involved in it. Denote by $[r_i]$ the units of the quantity r_i. Then we have

$$[r_i] = [r_o] = \frac{\text{Volume}}{\text{Time}}, \quad [q_i] = [q_o] = \frac{\text{Mass}}{\text{Volume}},$$

$$[V] = \text{Volume}, \quad [Q] = \text{Mass}.$$

Definition 1.5.4. The ***Water Tank Problem*** refers to water coming into a tank at a rate r_i with salt concentration q_i, and going out the tank at a rate r_o and salt concentration q_o, so that the water volume V and the total amount of salt Q, which is ______________________________, in the tank satisfy the equations,

$$V'(t) = r_i(t) - r_o(t), \tag{1.5.1}$$

$$Q'(t) = r_i(t)\, q_i(t) - r_o(t), q_o(t), \tag{1.5.2}$$

$$q_o(t) = \frac{Q(t)}{V(t)}, \tag{1.5.3}$$

$$r_i'(t) = r_o'(t) = 0. \tag{1.5.4}$$

Theorem 1.5.3. The amount of salt Q in a water tank problem defined in Def. 1.5.4 satisfies the differential equation

$$\underline{\hspace{8cm}} \tag{1.5.5}$$

where the coefficients in the equation are given by

$$a(t) = -\frac{r_o}{(r_i - r_o)\,t + V_0}, \qquad b(t) = r_i\, q_i(t). \tag{1.5.6}$$

Proof of Theorem 1.5.3:

□

Example 1.5.3: Consider a water tank problem with water rates $r_i = r_o = r = 2$ liters/min, fresh water is coming into the tank, the initial water volume in the tank is $V_0 = 200$ liters, and the initial salt in the tank is $Q_0 = 200$ grams. Then, find the time t_1 such that the salt in the tank is 1% the initial value.

Solution:

$\lhd$

1.5.4. **Exercises.**

1.5.1.- A radioactive material decays at a rate proportional to the amount present. Initially there are 50 milligrams of the material present and after one hour the material has lost 80% of its original mass.

(a) Find the mass of the material as function of time.
(b) Find the mass of the material after four hours.
(c) Find the half-life of the material.

1.5.2.- A vessel with liquid at 18 C is placed in a cooler held at 3 C, and after 3 minutes the temperature drops to 13 C.

(a) Find the differential equation satisfied by the temperature T of a liquid in the cooler at time $t = 0$.
(b) Find the function temperature of the liquid once it is put in the cooler.
(c) Find the liquid cooling constant.

1.5.3.- A tank initially contains $V_0 = 100$ liters of water with $Q_0 = 25$ grams of salt. The tank is rinsed with fresh water flowing in at a rate of $r_i = 5$ liters per minute and leaving the tank at the same rate. The water in the tank is well-stirred. Find the time such that the amount the salt in the tank is $Q_1 = 5$ grams.

1.5.4.- A tank initially contains $V_0 = 100$ liters of pure water. Water enters the tank at a rate of $r_i = 2$ liters per minute with a salt concentration of $q_1 = 3$ grams per liter. The instantaneously mixed mixture leaves the tank at the same rate it enters the tank. Find the salt concentration in the tank at any time $t \geqslant 0$. Also find the limiting amount of salt in the tank in the limit $t \to \infty$.

1.5.5.- A tank with a capacity of $V_m = 500$ liters originally contains $V_0 = 200$ liters of water with $Q_0 = 100$ grams of salt in solution. Water containing salt with concentration of $q_i = 1$ gram per liter is poured in at a rate of $r_i = 3$ liters per minute. The well-stirred water is allowed to pour out the tank at a rate of $r_o = 2$ liters per minute. Find the salt concentration in the tank at the time when the tank is about to overflow. Compare this concentration with the limiting concentration at infinity time if the tank had infinity capacity.