1.4. EXACT DIFFERENTIAL EQUATIONS

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1.4.1. Exact Equations.

**Definition 1.4.1.** An *exact* differential equation has the form

\[
\frac{dy}{dx} = \frac{M(x,y)}{N(x,y)},
\]

where the functions \( N \) and \( M \) satisfy

\[
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.
\]

**Remark:** Functions \( \text{__________} \) depend on \( \text{__________} \), and use the notation

\[
\text{\__________}.
\]

**Example 1.4.1:** Show whether a separable equation \( h(y) \frac{dy}{dt} = g(t) \) is exact or not.

**Solution:**

\[
\text{\__________}.$$
Example 1.4.2: Show whether the linear differential equation below is exact or not,

\[ y' = a(t) y + b(t), \quad a(t) \neq 0. \]

Solution:

Example 1.4.3: Show whether \(2ty' + 2t + y^2 = 0\) is exact or not.

Solution:
1.4.2. Solving Exact Equations.

**Remark:** Exact equations can be transformed into a total derivative, hence simple to solve.

**Theorem 1.4.2. (Exact Equations)** If the differential equation

\[
\frac{\partial}{\partial r} \psi + \frac{\partial}{\partial \theta} \psi = 0
\]

is exact, then it can be written as

\[
\frac{\partial}{\partial r} \psi + \frac{\partial}{\partial \theta} \psi = 0,
\]

where ______ is a potential function satisfying

\[
\frac{\partial}{\partial \theta} \psi = 0.
\]

Therefore, the solutions of the exact equation are given in implicit form as

\[
\frac{\partial}{\partial r} \psi = 0.
\]

**Remark:** The proof of the theorem above needs the following result.

**Theorem 1.4.3. (Poincaré)** Continuously differentiable functions \(M, N\) satisfy

\[
\frac{\partial}{\partial r} \psi + \frac{\partial}{\partial \theta} \psi = 0
\]

iff there exists a function ______, called potential function, such that

\[
\frac{\partial}{\partial \theta} \psi = 0.
\]

**Proof of Poincaré Theorem 1.4.3:**

\(\Rightarrow\) It is hard. Poincaré 1880.

\(\Leftarrow\)
Proof of Theorem 1.4.2:
Example 1.4.6: Find all solutions $y$ to the differential equation

$$2ty' + 2t + y^2 = 0.$$ 

Solution:
1.4.3. Semi-Exact Equations.

**Definition 1.4.4.** A *semi-exact* differential equation is a ______________ 
equation that can be transformed into an ____________ equation after a multipli-
cation by an integrating factor.

**Example 1.4.8:** Show that linear differential equations \( y' = a(t) y + b(t) \) are semi-exact.

**Solution:**
1.4.4. Solving Semi-Exact Equations.

**Theorem 1.4.5.** If the equation \( N y' + M = 0 \) is not exact, with \( \partial_t N \neq \partial_y M \), the function \( N \neq 0 \), and where the function \( h \) defined as

\[
\text{depending only on } \text{, then the equation below is exact}
\]

\[
\text{where } \text{is an antiderivative of } \text{.}
\]

**Remarks:**

(a) The function \( \mu(t) = e^{H(t)} \) is called an ________________________.

(b) Any ________________________ is solution of the differential equation

\[
\text{transforming a non-exact equation}
\]

(c) Multiplication by an ________________________ transforms a non-exact equation

\[
\text{into an exact equation.}
\]

This is exactly what happened with linear equations.

**Verification Proof of Theorem 1.4.5:**
Constructive Proof of Theorem 1.4.5:
**Example 1.4.9:** Find all solutions $y$ to the differential equation

$$ (t^2 + t y) y' + (3t y + y^2) = 0. \quad (1.4.5) $$

**Solution:**
1.4.5. **The Equation for the Inverse Function.**

**Remark:** We change the notation in this last part of the section.

(a) We change the independent variable name from \( t \) to \( x \).

(b) We write a differential equation as

\[
\text{...}
\]

(c) \( x(y) \) is the inverse of \( y(x) \), that is, \( x(y_1) = x_1 \iff y(x_1) = y_1 \).

(d) Recall \( x'(y) = \frac{1}{y'(x)} \).

**Remark:** But for exact equations it makes no difference to solve for \( y \) or its inverse \( x \).

**Theorem 1.4.6.** The equation \( \phantom{=} \) is exact

iff the equation \( \phantom{=} \) is exact.

**Remark:** For non-exact equations there is a difference.

**Proof of Theorem 1.4.6:**

\[\square\]
1.4.6. **Solving for the Inverse Function.**

**Remark:** Sometimes the equations $N y' + M = 0$ and $N + M x' = 0$ are written together,

This equation deserves two comments:

(a) We do not use this notation here. That equation makes sense in the framework of differential forms, which is beyond the subject of these notes.

(b) Some people justify the use of that equation outside the framework of differential forms by thinking $y' = \frac{dy}{dx}$ as real fraction and multiplying $N y' + M = 0$ by the denominator,

$$N \frac{dy}{dx} + M = 0 \Rightarrow N \frac{dy}{dx} + M \frac{dx}{dx} = 0.$$ 

Unfortunately, $y'$ is not a fraction $\frac{dy}{dx}$, so the calculation just mentioned has no meaning.

**Theorem 1.4.7.** If the equation $M x' + N = 0$ is not exact, with $\partial_y M \neq \partial_x N$, the function $M \neq 0$, and where the function $\ell$ defined as

depends only on __________________________, then the equation below is exact,

______________________________

where $L$ is an antiderivative of $\ell$.

**Remarks:**

(a) The function $\mu(y) = e^{L(y)}$ is called an __________________________.

(b) Any __________________________ is solution of the differential equation

______________________________

c) Multiplication by an __________________________ transforms a non-exact equation

______________________________ into an exact equation.

______________________________

This is exactly what happened with linear equations.
Verification Proof of Theorem 1.4.7:
Constructive Proof of Theorem 1.4.7:
**Example 1.4.11:** Find all solutions to the differential equation

\[(5x e^{-y} + 2 \cos(3x)) y' + (5 e^{-y} - 3 \sin(3x)) = 0.\]

**Solution:**