1.4. Exact Differential Equations

Section Objective(s):

- Exact Equations.
- Solving Exact Equations.
- Semi-Exact Equations.
- Solving Semi-Exact Equations.
- The Equation for the Inverse Function.
- Solving for the Inverse Function.

1.4.1. **Exact Equations.**

Definition 1.4.1. An ***exact*** differential equation has the form

____,

where the functions N and M satisfy

____.

Remark: Functions ______________ depend on __________, and use the notation

____.

Example 1.4.1: Show whether a separable equation $h(y)\, y'(t) = g(t)$ is exact or not.

Solution:

$\triangleleft$

Example 1.4.2: Show whether the linear differential equation below is exact or not,

$$y' = a(t)\, y + b(t), \qquad a(t) \neq 0.$$

Solution:

◁

Example 1.4.3: Show whether $2ty\, y' + 2t + y^2 = 0$ is exact or not.

Solution:

◁

1.4.2. Solving Exact Equations.

Remark: Exact equations can be transformed into a total derivative, hence simple to solve.

Theorem 1.4.2. (Exact Equations) If the differential equation

____

is exact, then it can be written as

____,

where _________ is a potential function satisfying

____.

Therefore, the solutions of the exact equation are given in implicit form as

____.

Remark: The proof of the theorem above needs the following result.

Theorem 1.4.3. (Poincaré) Continuously differentiable functions M, N satisfy

____

iff there exists a function _________, called potential function, such that

____.

Proof of Poincaré Theorem 1.4.3:

($\Rightarrow$) It is hard. Poincaré 1880.

($\Leftarrow$)

$\square$

Proof of Theorem 1.4.2:

□

EXAMPLE 1.4.6: Find all solutions y to the differential equation

$$2ty\,y' + 2t + y^2 = 0.$$

SOLUTION:

◁

1.4.3. **Semi-Exact Equations.**

> **Definition 1.4.4.** A ***semi-exact*** differential equation is a ________________________ equation that can be transformed into an ________________ equation after a multiplication by an integrating factor.

EXAMPLE 1.4.8: Show that linear differential equations $y' = a(t)\,y + b(t)$ are semi-exact.

SOLUTION:

$\lhd$

1.4.4. **Solving Semi-Exact Equations.**

Theorem 1.4.5. If the equation $N\,y' + M = 0$ ______ is *not exact*, with $\partial_t N \neq \partial_y M$, the function $N \neq 0$, and where the function h defined as

______

depends only on ______, then the equation below is exact

______,

where ______ is an antiderivative of ______,

______.

Remarks:

(a) The function $\mu(t) = e^{H(t)}$ is called an ______.

(b) Any ______ is solution of the differential equation

______.

(c) Multiplication by an ______ transforms a non-exact equation

______

into an exact equation.

______.

This is exactly what happened with linear equations.

Verification Proof of Theorem 1.4.5:

Constructive Proof of Theorem 1.4.5:

□

Example 1.4.9: Find all solutions y to the differential equation

$$\big(t^2 + t\,y\big)\,y' + \big(3t\,y + y^2\big) = 0. \tag{1.4.5}$$

Solution:

◁

1.4.5. **The Equation for the Inverse Function.**

Remark: We change the notation in this last part of the section.

(a) We change the independent variable name from t to x.
(b) We write a differential equation as

____.

(c) $x(y)$ is the inverse of $y(x)$, that is, $x(y_1) = x_1 \quad \Leftrightarrow \quad y(x_1) = y_1$.
(d) Recall $x'(y) = \dfrac{1}{y'(x)}$.

Remark: But for exact equations it makes no difference to solve for y or its inverse x.

Theorem 1.4.6. The equation ________________________ is exact

iff the equation ________________________ is exact.

Remark: For non-exact equations there is a difference.

Proof of Theorem 1.4.6:

$\square$

1.4.6. **Solving for the Inverse Function.**

Remark: Sometimes the equations $N\,y' + M = 0$ and $N + M\,x' = 0$ are written together,

____.

This equation deserves two comments:

(a) We do not use this notation here. That equation makes sense in the framework of differential forms, which is beyond the subject of these notes.

(b) Some people justify the use of that equation outside the framework of differential forms by thinking $y' = \dfrac{dy}{dx}$ as real fraction and multiplying $N\,y' + M = 0$ by the denominator,

$$N\,\frac{dy}{dx} + M = 0 \quad \Rightarrow \quad N\,dy + M\,dx = 0.$$

Unfortunately, y' is not a fraction $\dfrac{dy}{dx}$, so the calculation just mentioned has no meaning.

Theorem 1.4.7. If the equation $M\,x' + N = 0$ ________________ is *not exact*, with $\partial_y M \neq \partial_x N$, the function $M \neq 0$, and where the functionℓ defined as

____

depends only on ________________________, then the equation below is exact,

____,

where L is an antiderivative of ℓ,

____.

Remarks:

(a) The function $\mu(y) = e^{L(y)}$ is called an __________________________________.

(b) Any ____ is solution of the differential equation

________________________________.

(c) Multiplication by an __________________________________ transforms a non-exact equation

____

into an exact equation.

____.

This is exactly what happened with linear equations.

Verification Proof of Theorem 1.4.7:

□

Constructive Proof of Theorem 1.4.7:

□

Example 1.4.11: Find all solutions to the differential equation

$$\big(5x\, e^{-y} + 2\cos(3x)\big)\, y' + \big(5\, e^{-y} - 3\sin(3x)\big) = 0.$$

Solution: