1.1. Linear Constant Coefficient Equations

Section Objective(s):
- Overview of Differential Equations.
- Linear Differential Equations.
- Solving Linear Differential Equations.
- The Initial Value Problem.

1.1.1. **Overview of Differential Equations.**

Remark: A differential equation is ________________________, the unknown is ______________________, and both ____ may appear in the equation.

Example 1.1.1:

(a) **Newton's Law:** Mass times acceleration equals force, $ma = f$, where m is the particle mass, $a = d^2x/dt^2$ is the particle acceleration, and f is the force acting on the particle. Hence Newton's law is the differential equation

____.

where the unknown is the position of the particle in space, $\boldsymbol{x}(t)$, at the time t.

Remark: This is a second order Ordinary Differential Equation (ODE).

(b) **Radioactive Decay:** The amount u of a radioactive material changes in time as follows,

____________________________________, $\quad k > 0,$

where k is a positive constant representing radioactive properties of the material.

Remark: This is a first order ODE.

(c) **The Heat Equation:** The temperature T in a solid material changes in time and in one space dimension according to the equation

____, $\quad k > 0,$

where k is a positive constant representing thermal properties of the material.

Remark: This is a first order in time and second order in space PDE.

(d) **The Wave Equation:** A wave perturbation u propagating in time t and in one space dimension x through the media with wave speed $v > 0$ is

____.

Remark: This is a second order in time and space Partial Differential Equation (PDE).

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1.1.2. **Linear Differential Equations.**

Definition 1.1.1. A ***first order ODE*** on the unknown y is

$$\underline{\hspace{8cm}}, \tag{1.1.1}$$

where f is given and $y' = \dfrac{dy}{dt}$. The equation is __________________ iff the source function f is linear on its second argument,

$$\underline{\hspace{8cm}}. \tag{1.1.2}$$

The linear equation has ________________________ coefficients iff both a and b above are constants. Otherwise the equation has _______________________ coefficients.

Example 1.1.2:

(a) $y' = 2\,y + 3$ is ____.

(b) $y' = -\dfrac{2}{t}\,y + 4t$ is ____.

(c) $y' = -\dfrac{2}{t}\,\dfrac{1}{y} + 4t$ is ____.

$\triangleleft$

Example 1.1.3: Show that $y(t) = e^{2t} - \dfrac{3}{2}$ is solution of the equation $y' = 2\,y + 3$.

Solution:

$\triangleleft$

1.1.3. **Solving Linear Differential Equations.**

Theorem 1.1.2. (Constant Coefficients) The linear differential equation

$$\underline{\hspace{6cm}}, \tag{1.1.3}$$

with $a \neq 0$, b constants, has infinitely many solutions,

$$\underline{\hspace{8cm}}. \tag{1.1.4}$$

Remark: Equation (1.1.4) is called the *general solution* of the differential equation in (1.1.3).

Proof of Theorem 1.1.2:

$\square$

EXAMPLE 1.1.5: Find all solutions to the constant coefficient equation $y' = 2y + 3$.

SOLUTION:

$\triangleleft$

1.1.4. **The Initial Value Problem.**

Definition 1.1.3. The ***initial value problem*** (IVP) is to find all solutions y to

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_, \tag{1.1.5}$$

that satisfy the initial condition

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_, \tag{1.1.6}$$

where a, b, and y_0 are given constants.

Remark: The differential equation $y' = a\,y + b$ has infinitely many solutions, but the associated IVP has only one solution.

Theorem 1.1.4. (Constant Coefficients IVP) The initial value problem

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_,$$

for given constants $a, b, y_0 \in \mathbb{R}$, and $a \neq 0$, has the unique solution

$$\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_. \tag{1.1.7}$$

Proof of Theorem 1.1.4:

$\square$

EXAMPLE 1.1.8: Find the unique solution of the initial value problem

$$y' = 2y + 3, \qquad y(0) = 1. \tag{1.1.8}$$

SOLUTION:

$\triangleleft$