Springs as First Order Systems

The mass-spring equation as a first order linear differential system

Team Member: 1. ______________________

Objectives

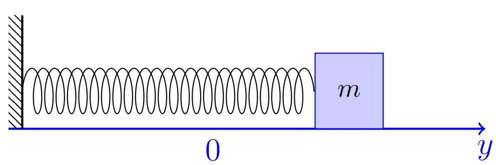
To have a better understanding of direction fields, phase portraits, and graph of solution functions of a first order linear system of differential equations.

Introduction

A mass-spring system consists of an object attached to a spring and sliding on a table. We assume that:

- the object has mass $m > 0$,
- the spring has spring constant $k > 0$,
- the friction with the table produces a damping force with damping constant $d \geqslant 0$.

We denote by $y(t)$ the displacement of the mass as a function of time, where $y = 0$ represents the rest position of the mass.



The damping force is proportional to the object's velocity, acting in the opposite direction, and it is given by $f_d = -d\, y'$. Therefore, Newton's equation of motion in this case is

$$m\, y''(t) + d\, y'(t) + k\, y(t) = 0. \tag{1}$$

Question 1. Consider the mass-spring second order equation

$$y'' + \frac{d}{m}\,y' + \frac{k}{m}\,y = 0. \tag{2}$$

(1a) *(1/3 points)* Find the matrix A such that the equation above for the mass-spring can be written as the first order system

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \text{where} \qquad \boldsymbol{x} = \begin{bmatrix} x_1 = y \\ x_2 = y' \end{bmatrix}.$$

(1b) *(1/3 points)* Compute the characteristic equation for equation (2) and the characteristic equation of the 2×2 matrix A. How are these quantities related?

(1c) *(1/3 points)* Find a formula for the eigenvalues of the coefficient matrix found above in terms of m, d, and k.

The **MathStudio** interactive graph below shows different aspects of the solution to the mass-spring system. The interactive graph has two parts:

- In the top part it shows the graph of the position function $x_1(t)$, in purple, and the velocity function $x_2(t)$, in blue, for the mass-spring system. Click on the buttons **X1-Purple** and **X2-Blue** to turn them on or off.
- In the bottom part the interactive graphs shows the direction field, in blue, and the solution curves, in red, for the first order reduction of the mass-spring system. Click on the button **Sol-Curve-Red** to turn the solution curve on or off.
- The slider **Time** highlights with a green dot the solution values at that time.
- The sliders **X1-0** and **X2-0** are the initial conditions for the functions x_1 and $x - 2$, respectively.
- The sliders **k-over-m** and **d-over-m** give values to the constants $\frac{k}{m}$ and $\frac{d}{m}$, respectively.

Mass-Spring Interactive Graph.

Question 2. Consider a mass-spring with parameters

$$d = 0, \quad \frac{k}{m} = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

(2a) *(1/2 points)* Do the position and velocity functions oscillate in time? Is the amplitude of the oscillations constant, increasing, or decreasing in time? Is the behavior of the solutions consistent with no friction, with small friction, or with large friction?

(2b) *(1/2 points)* Are the position and velocity functions oscillating in phase or out of phase with each other? If they are out of phase, what is the approximate phase shift? Also, compare their amplitudes.

Question 3. Consider a mass-spring with parameters as in **Question 2** except $\frac{k}{m}$, which is now set to

$$\frac{k}{m} = 4.$$

(3a) *(1 point)* Find the eigenvalues of the first order reduction of the mass-spring system in both cases.

(3b) *(1 point)* How do the position and velocity functions change when we change $\frac{k}{m}$ from 1 to 4? Describe the change in their wavelengths and in their amplitudes.

(3c) *(1 point)* Give a physical interpretation for the change in the wavelengths and amplitudes described above. That is, compare the movement of the actual springs with $\frac{k}{m} = 1$ and $\frac{k}{m} = 4$.

Question 4. Consider a mass-spring with parameters

$$\frac{d}{m} = 0.5, \quad \frac{k}{m} = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

(4a) *(1 point)* Find the eigenvalues of the first order reduction of the mass-spring system in this case.

(4b) *(1 point)* Do the position and velocity functions oscillate in time? Is the amplitude of the oscillations constant, increasing, or decreasing in time? Is the system going to stop in a finite or an infinite time? Is the behavior of the solutions consistent with no friction, with small friction, or with large friction?

(4c) *(1 point)* Is the solution curve for this case a closed curve or an open curve? Compare with the systems studied in **Questions 2 and 3**. Explain what is the reason for this change in the solution curve, from closed to open.

Question 5. Consider a mass-spring with parameters

$$\frac{d}{m} = 2, \quad \frac{k}{m} = 1, \quad y(0) = 1, \quad y'(0) = 0.$$

(5a) *(2/3 points)* Find the eigenvalues of the first order reduction of the mass-spring system in this case.

(5b) *(2/3 points)* Do the position and velocity functions oscillate in time? Is the amplitude of the oscillations constant, increasing, or decreasing in time? Is the system going to stop in a finite or an infinite time? Is the behavior of the solutions consistent with no friction, with small friction, or with large friction?

(5c) *(2/3 points)* Look back at the answers of **Questions 2, 3, 4, and 5a**, and then relate the eigenvalues of the first order reduction for the mass-spring system with the behavior of its solutions. In particular, pure imaginary, complex with real part negative, and real, with oscillations, decay, and no oscillations of the solution.