Nonlinear Pendulum

The equation for a pendulum as a first order differential system

Team Members: 1. ____________________

2. ____________________

3. ____________________

4. ____________________

Objectives

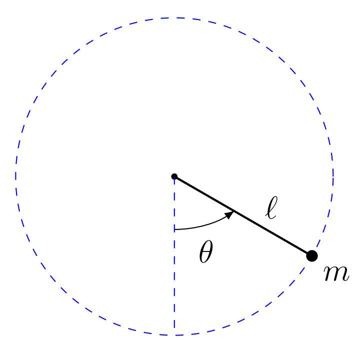
To have a better understanding of direction fields, phase portraits, and graph of solutions functions of the equations describing a nonlinear pendulum.

Introduction

A pendulum consists of a small ball attached to the end of a rigid rod, the latter can swing from its other end, as shown in the picture below. We assume that the ball at the end of the pendulum has mass $m > 0$ and the pendulum rod has length $\ell > 0$. Also assume that the there is a friction force acting on the ball, with damping coefficient $d > 0$. Recall that a friction force opposes the movement and is proportional to the speed of the object.

The mass at the end of the pendulum will move on a circle of radius ℓ centered at the oscillation point. Our main variable is $\theta(t)$, the angle between the rod and the downward vertical, positive in the counter clockwise direction. The displacement of the pendulum mass along the the circle where it moves is $s(t) = \ell\,\theta(t)$. Newton's equation of motion is

$$m\,(\ell\theta)'' = -mg\sin(\theta) - d\,(\ell\theta)'.$$



The negative sign in the first term of the right-hand side above is because the component of the weight tangent to the circle is in the direction opposite to the direction of increasing θ. The negative in the second term above is because the friction force opposes the movement. We rewrite the equation as

$$\theta'' = -\frac{g}{\ell}\sin(\theta) - \frac{d}{m}\,\theta'.$$

In this Lab we will focus on the case $g/\ell = 1$ and $m = 1$, that is

$$\theta'' = -\sin(\theta) - d\,\theta'.$$

Critical Points and Linearization: No Friction

Question 1. *(3 points)* Consider the case of no friction, $d = 0$.

(1a) Write the equation for the pendulum as a first order system for

$$u(t) = \theta(t), \qquad v(t) = \theta'(t).$$

(1b) Find the equilibrium points of the first order system found in Question 1.

(1c) Linearize the system at each of the equilibrium points and determine their stability type.

Graphical Analysis: No Friction Case

In the interactive graph below we show the direction field and solution curves for the first order reduction of the nonlinear pendulum in the case $g/\ell = 1$ and $m = 1$, with $0 \leqslant d < 2$.

Interactive Graph Link.

Question 2. *(3 points)* We consider the case of no friction. Use the interactive graph above with $d = 0$ and answer the following:

(2a) Plot several solution curves and describe the two main kinds of qualitatively different solution curves.

(2b) What is the position of the pendulum in each type of critical point?

(2c) What is the physical interpretation of each type of solution curve in terms of motion of the pendulum.

Question 3. *(2 points)* Use the interactive graph above with $d = 0$ and/or your physical intuition to answer the following:

(3a) An initial condition which makes the pendulum make oscillations of up to $\pi/4$ around the lowest position of the ball,

(3b) What is the initial velocity such that the pendulum starts at $\theta = 0$ and reaches the unstable critical point $(\pi, 0)$ at infinite time? Can one obtain experimentally this solution?

Graphical Analysis: Small Friction Case

In the case of small friction, $0 < d < 2$, we get $d^2 - 4 < 0$, and one can find the following:

- For even critical points,

$$\lambda_{e\pm} = \frac{1}{2}(-d \pm i\sqrt{4-d^2}) \quad \Rightarrow \quad ((2k)\pi, 0) \quad \text{are } \textbf{stable spirals}.$$

- For odd critical points,

$$\lambda_{o\pm} = \frac{1}{2}(-d \pm \sqrt{d^2+4}) \quad \Rightarrow \quad \lambda_{o-} < 0 < \lambda_{o+} \quad \Rightarrow \quad ((2k+1)\pi, 0) \quad \text{are } \textbf{saddle nodes}.$$

Question 4. *(2 points)* Use the interactive graph above with small friction, say $d \simeq 0.2$ and look at the solution curves in this case for different initial conditions.

(4a) Is there an initial condition which makes the pendulum go round and round in full circles (infinitely many times)? Explain based on the graph and based on the physics of the problem.

(4b) What happens as the friction gets larger and larger?