Modeling Infectious Diseases

The SIR (Susceptible, Infected, Recovered) Epidemic Model

Team Members: 1. ______________________

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Objectives

To construct and interpret models using systems of ordinary differential equations in various settings.

Introduction

We consider a classical epidemic model and we used to describe a chickenpox infection. We will:

- Interpret the terms in a system of differential equations.
- Find equilibrium solutions and interpret them in the setting of the model.
- Study the behavior of equilibrium solutions as the parameters of the system vary. Equilibrium solutions are identified on the phase-portrait graph and on the component plot.
- Construct a new model that incorporates vaccination and analyze how vaccination changes the long-term behavior of solutions.

SIR Epidemic Model

Suppose we have a disease such as chickenpox, which, after recovery, provides immunity. In this model we will assume the number of individuals is constant, $N > 0$. Suppose that the disease is such that the population can be divided into three distinct classes:

- The **susceptible** people, S, who can catch the disease.
- The **infected** people, I, who have the disease and can transmit it.
- The **recovered** people, R, who have had the disease and are recovered and immune to it.

Since the total number of individuals is constant,

$$S(t) + I(t) + R(t) = N.$$

where $S(t)$, $I(t)$, and $R(t)$ represent the number of individuals at time t in each class, respectively.

We can represent how individuals move from one category to another in the following way:

$$S \xrightarrow[\beta]{} I \xrightarrow[\gamma]{} R$$

where $\beta > 0$ is the **infection rate**, and $\gamma > 0$ is the **recovery rate**. We assume that the individuals in all classes are well-mixed, i.e., every pair of individuals has an equal probability of coming into contact with one another. In this case, our system can be described by a system of first order differential equations for the variables $S(t)$, $I(t)$, $R(t)$, called the **SIR Model**, given by

$$S' = -\beta\, S\, I \tag{1}$$

$$I' = \beta\, S\, I - \gamma\, I \tag{2}$$

$$R' = \gamma\, I \tag{3}$$

The physical interpretation of the different terms in each equation is the following.

- Equation (1) states that the number of susceptible individuals decreases at a rate proportional to the number of susceptible individuals times the infected individuals. This product is proportional to the number of encounters between susceptible and infected individuals.
- All the susceptible individuals who became infected move to the class of infected people, therefore, in equation (2) we have that the number of infected individuals increases at the same rate that the number of susceptible individuals decreases—the first term. The second term in equation (2) represents the number of infected individuals decreasing because of the people who are recovering and moving into the class of recovered individuals. This number is proportional to the number of infected peolpe, with a constant of proportionality, γ—the recovery rate.
- Equation (3) says that the rate of change of the recovered individuals, R', is proportional to the number of infected individuals, I, where the proportionality constant is the recovery rate, γ. This equation means that for a given unit of time, a proportion γ of the infected individuals move to the class of recovered individuals.

Question 1. *(1 point)* Show that the equations (1)-(3) are consistent with the hypothesis that the total number of individuals is constant in time.

Question 2. *(1 point)* Find the equilibrium solutions of the SIR Model.

We now introduce an interactive graph using Math Studio. This graph shows the functions S, I, and R, solutions of the SIR Model for particular values of the parameters: β, the **Infection-Rate**; γ, the **Recovery-Rate**; and **Initial-Infected**, $I(0)$, the initial value of the infected individuals. The functions in the graph are color-coded and can be turned on or off by clicking on the corresponding buttons, as follows.

- The susceptible individuals S, as function of time, is controlled by the button **Susceptible-Blue**.
- The Infected individuals I, as function of time, is controlled by the button **Infected-Red**.
- The recovered individuals R, as function of time, is controlled by the button **Recovered-Green**.
- We also show a phase portrait, where the horizontal axis represents the susceptible individuals, S, and the vertical axis represents the infected individuals, I.
- The solution curve in the phase portrait is controlled by the button **Sol-Curve-Purple**.

Below is the link to the Math Studio interactive graph.

SIR Model with-without Vaccination

Question 3. *(3 points)*

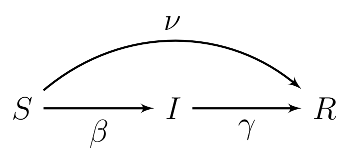
(3a) Set the system parameters to: **Initial-Infected** $I(0) = 0.1$, **Infection-Rate** $\beta = 0.5$, and **Recovery-Rate** $\gamma = 0.5$. Use the solution graphs and/or the solution curve for this case, to find the equilibrium solution of the system. What is the total population in this system?

(3b) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.3$, and the recovery rate $\gamma = 0.8$?

(3c) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.4$, and the recovery rate $\gamma = 0.8$?

SIR Epidemic Model with Vaccination

Now, assume we have the same model as above, but a vaccine was invented against the disease. The vaccine sends some of the susceptible individual directly to the recovered individuals, since they are now immune to the disease. We can represent how individuals move from one category to another in the following way:



Question 4. *(2 points)* Assume that a **fixed proportion**, ν, of susceptible people are being vaccinated per time period. Modify the model given in equations (1)-(3) to account for the vaccination.

Question 5. *(2 points)* Find the equilibrium solutions for the SIR model with vaccination.

Hint: Since $\nu > 0$, then $(-\beta I - \nu) \neq 0$.

Question 6. *(1 point)* Set the system parameters to: **Initial-Infected** $I(0) = 0.5$, **Infection-Rate** $\beta = 0.3$, and **Recovery-Rate** $\gamma = 0.8$. Also set the **Vaccination** rate $\nu = 0.2$. Use the solution graphs and/or the solution curve for this case, to find the equilibrium solution of the system. Compare this result with your answer to question **(3b)**.