Linear Differential Systems

The mass-spring equation as a first order linear differential system

Team Members: 1. ______________________

2. ______________________

3. ______________________

4. ______________________

Objectives

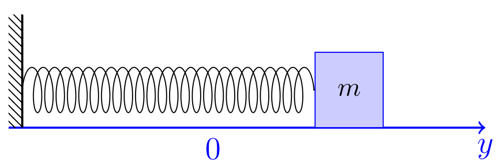
To have a better understanding of direction fields, phase portraits, and graph of solutions functions of a first order linear system of differential equations.

Introduction

A mass-spring system is an object attached to a spring and slides on a table. The system has the following properties:

- the object has mass $m > 0$,
- the spring has spring constant $k > 0$,
- the friction with the table produces a damping force with damping constant $d \geqslant 0$.

As usual, we denote by $y(t)$ the displacement of the mass as a function of time, where $y = 0$ represents the rest position of the mass.



The damping force is proportional to the object's velocity, acting in the opposite direction, and it is given by $f_d = -d\,y'$. Therefore, Newton's equation of motion in this case is

$$m\,y''(t) + d\,y'(t) + k\,y(t) = 0. \tag{1}$$

Question 1.*(1 point)* Show that the characteristic equation for the mass-spring second order equation

$$m\,y'' + d\,y' + k\,y = 0$$

is the same as the characteristic equation of the 2×2 matrix A, the coefficient matrix of the first order reduction of the mass-spring equation,

$$\boldsymbol{x}' = A\boldsymbol{x}, \qquad \boldsymbol{x} = \begin{bmatrix} y \\ y' \end{bmatrix}.$$

The interactive graph below shows different aspects of the solution to the mass-spring system. The interactive graph has two parts. In the top part it shows the graph of the position and velocity solutions of the mass-spring system as functions of time. In the bottom part the interactive graphs shows the direction field and the solution curves for the first order reduction of the mass-spring system.

Interactive Graph Link.

Question 2. For **each of the three situations below** do the following:

(2a) *(1 point per situation)* Find the eigenvalues of the coefficient matrix of the first order reduction of the mass-spring system.

(2b) *(1 point per situation)* Use the given interactive graph to make a sketch the solution curve in the yy'-plane. Also make a sketch of the position, $y(t)$, and velocity, $y'(t)$, of the mass attached to the spring as functions of t.

(2c) *(1 point per situation)*

- Do the position and velocity functions oscillate in time?
- Do the position and velocity functions decay in time or grow in time or neither?
- Is the solution curve a closed curve or an open curve?
- What is the limit $t \to \infty$ of the position function, velocity function, and solution curve?
- Can you change the initial data and find a solution of the mass-spring system with a closed curve **and** with a position function without oscillations in time?

Situation (i): Consider the case when there is **no friction** and $\dfrac{k}{m} = 4$.

Situation (ii): Consider the case when there is **small friction**, $\dfrac{d}{m} = 1$ and $\dfrac{k}{m} = 4$.

Situation (iii): Consider the case when there is **large friction**, $\dfrac{d}{m} = 4$ and $\dfrac{k}{m} = 4$.