Forced Oscillators, Beats, and Resonance

Team Members: 1. ______________________

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Objectives

Understanding the concepts of beats and resonance in simple oscillating systems.

Introduction

In this lab we will study the oscillations in a mass-spring system that is subject to an external force. We focus on two cases:

- First we study the oscillations of this system in the case that the frequency of the driver force is close—but not equal—to the natural frequency of the string-mass system.
- Then we study what happens when the external driver force has a frequency exactly equal to the natural frequency of the system.

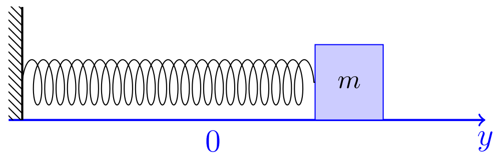
In the first case we will discover a particular type of behavior of the system, called beats. The oscillations of the system have a modulation in amplitude. This modulation has its own frequency, which depends on the difference between the driver and natural frequencies.

In the second case we discover the the behavior of the system called resonance. The amplitude of the oscillations grow without limit, and the physical system eventually breaks down. We also see how the beats solutions approximate the resonant solution.

The understanding of oscillations in simple systems—such as mass-spring systems—is the first step to understand vibrations in various engineering structures. The design of buildings and bridges must be such that resonance effects under the action of external forces—such as wind or earthquakes— are completely suppressed.

Part 1: Free Oscillations

A mass-spring system is a mass m attached to a spring with spring constant k that slides on a frictionless table. We denote by $y(t)$ the displacement of the mass as a function of time, where $y = 0$ represents the rest position of the mass.



This system is described by Newton's law of motion $ma = f$, where $a = y''$, and f is given by Hooke's law, $f(y) = -k\,y$. So the equation of motion for the oscillating spring without any external force is

$$m\,y'' + k\,y = 0.$$

The **natural frequency** of the oscillator is the frequency ω_0 of all oscillations when there is no external force acting on the system.

(1) *(1 point)* Consider now a spring-mass system with mass $m = 1$ grams, and $k = 25$ grams per second square. Find the general solution for this mass-spring system in case that there are no external forces, and find the **natural frequency** ω_0 of this system.

Part 2: Forced Oscillations

As before, assume that the spring-mass system has mass $m = 1$ grams, and $k = 25$ grams per second square. We now add an external force to the mass-spring system given by

$$f_{\text{ext}}(t) = \cos(\nu t), \qquad \nu > 0.$$

The frequency of the external force, ν, is called the **driver frequency**. The motion of a mass-spring system initially at rest and subject to the external force above is the solution of the initial value problem

$$y'' + 25\, y = \cos(\nu t), \quad \text{and} \quad y(0) = 0, \quad y'(0) = 0.$$

In the course website it is shown how to use the undetermined coefficients method to find the solution to this initial value problem, in the case that the driver frequency is different from the natural frequency of the system, $\nu \neq 5$. In this case the solution is

$$\boxed{y_{\text{NR}}(t) = \frac{1}{(25 - \nu^2)}\big(\cos(\nu t) - \cos(5t)\big)}, \qquad \nu \neq 5. \tag{1}$$

(2) *(2 points)* First, explain why we need the condition $\nu \neq 5$ in the expression (1). Second, keep time fixed in $y_{\text{NR}}(t)$ and use the L'Hopital rule in the variable ν to compute the limit

$$\lim_{\nu \to 5} y_{\text{NR}}(t). \qquad \text{(Recall, keeping } t \text{ fixed.)}$$

Part 3: Forced Resonant Oscillator

An oscillator is in **resonance** when the driver frequency ν of the external force is equal to the natural frequency $\omega_0 = 5$ of the unperturbed oscillator. The motion of a mass-spring system initially at rest and subject to the external force with driver frequency $\nu = 5$ is the solution of the initial value problem

$$y'' + 25\,y = \cos(5t), \qquad \text{and} \quad y(0) = 0, \quad y'(0) = 0. \tag{2}$$

In the course website it is shown how use the undetermined coefficients method to find the solution to this initial value problem, and the solution is

$$\boxed{y_{\rm R}(t) = \frac{t}{10}\sin(5t)}. \tag{3}$$

(3) *(1 point)* Verify that the function in Eq. (3) is solution of the resonant initial value problem in (2).

Note: You only need to introduce $y_{\rm R}$ in (3) into Eq. (2) and also verify the initial conditions.

Part 4: Graphical Analysis of the Forced Oscillations

We now give specific names to the functions found in the previous parts.

(a) Resonant case: $\nu = 5 = \omega_0$. In this case the solution to the initial value problem in (3) is

$$y_R(t) = \frac{t}{10}\sin(5t). \tag{4}$$

(b) Non-Resonant case: $\nu \neq 5$. In this case the solution to the initial value problem in (1) is

$$y_{NR}(t) = \frac{1}{(25-\nu^2)}\big(\cos(\nu t) - \cos(5t)\big). \tag{5}$$

Note: The Non-Resonant solution is the **sum** of two solutions, $y_{NR} = y_p + y_h$, where

$$y_p(t) = \frac{1}{(25-\nu^2)}\cos(\nu t), \qquad y_h(t) = -\frac{1}{(25-\nu^2)}\cos(5t). \tag{6}$$

We call y_p the **Particular** solution and y_h the **Homogeneous** solution.

(c) Note: An oscillatory function has **beats** when the function has a periodic modulation in amplitude with frequency smaller than the function frequency, as shown in the Figure 1.

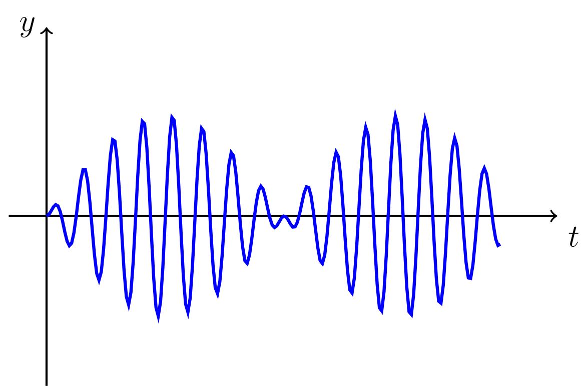


Figure 1: A function y showing the beats phenomena.

In order to answer the questions below, recall the solutions names given above:

- y_R in (4) is the **Resonant** solution;
- y_{NR} in (5) is the **Non-Resonant** solution;
- y_p and y_h in (6) are the **Particular** and **Homogeneous** solutions respectively.

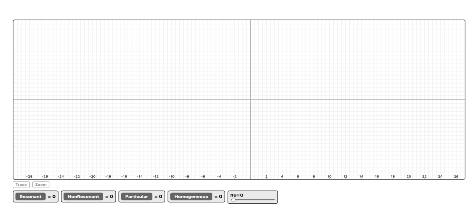


Figure 2: Picture of the interactive graph.

These solution names appear at the bottom of the interactive graph, which at the start will look as in Figure 2. When you click on these buttons in the interactive graph, the corresponding solution will show up, **Resonant** in blue, **Non-Resonant** in purple, **Particular** in red, and **Homogeneous** in green. The last button on the bottom right in Figure 2 is a slider that changes the value of the driver frequency ν in the interval $[0, 5]$, where 5 is the natural frequency of the system.

Now click on the following Interactive Graph Link:

Beats and Resonance in Mass-Springs.

(4.1) *(1 point)* In the Interactive Graph, turn on the **Non-Resonant**, keep off all the other functions, and vary the driver frequency ν from 0 to 5. Then, find the minimum value of the driver frequency ν such that the **Non-Resonant** solution displays beats.

(4.2) *(1 point)* We now try to understand why beats arises. Go back to the **interactive graph link** above. Keep the **Non-Resonant** solution on and set the driver frequency to $\nu = 4.7$, so the **Non-Resonant** solution displays beats. Now turn **on** both the **Particular** and the **Homogeneous** solutions. Relate the beats on the **Non-Resonant** solution with the values of the functions **Particular** and **Homogeneous**.

(4.3) *(1 point)* Describe how the **Non-Resonant** solution y_{NR} approaches the **Resonant** solution y_{R} as the driver frequency ν approaches the natural frequency $\omega_0 = 5$.

(4.4) *(1 point)* What is the behavior of the amplitude of the **Resonant** solution as the time variable $t \to \infty$? What will happen to the actual spring of a resonant system when time is large enough?

Part 5: Sound Waves

(5.1) *(1 point)* If the ODE described sounds waves, then what would the beats phenomenon sound like?

(5.2) *(1 point)* Watch the following short video: `https://www.youtube.com/watch?v=V8W4Djz6jnY`. What are the frequencies of the two tuning forks? Explain why beats occurs.