Modeling Using Systems of ODEs

Applications to Infectious Diseases

Team Members: 1. ______________________

2. ______________________

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Objectives

Learning to interpret and construct models using systems of ordinary differential equations in a variety of settings.

Introduction

In this lab we will consider a classical epidemic model and interpret it in the setting of chickenpox. We will:

- interpret the terms in a system of differential equations.
- find equilibrium solutions and interpret them in the setting of the model.
- study the behavior of equilibrium solutions as the parameters of the system vary. Equilibrium solutions are identified on the phase-portrait graph and on the component polot.
- construct two model,s which incorporate vaccination and analyze how vaccination changes the long-term behavior of solutions.

SIR Epidemic Model

Suppose we have a disease (like chickenpox) which, after recovery, provides immunity. In this model we will assume the number of individuals is constant. Suppose that the disease is such that the population can be divided into three distinct classes:

1. The susceptibles, S, who can catch the disease;
2. The infectives, I, who have the disease and can transmit it;
3. The recovered, R, who have had the disease and are recovered and immune to it.

We can represent how individuals move from one category to another in the following way:

$$S \xrightarrow[\beta]{} I \xrightarrow[\gamma]{} R$$

where $S(t)$, $I(t)$, and $R(t)$ represent the number of individuals at time t in each class, respectively, $\beta > 0$ is the **infection rate**, and $\gamma > 0$ is the **recovery rate**.

We assume that the individuals in all classes are well-mixed, i.e., every pair of individuals has an equal probability of coming into contact with one another. In this case, our SIR model has the following form:

$$\frac{dS}{dt} = -\beta S\, I \tag{1}$$

$$\frac{dI}{dt} = \beta S\, I - \gamma I \tag{2}$$

$$\frac{dR}{dt} = \gamma I \tag{3}$$

Question 1. Explain each of the terms in equations (1) and (2).
(For example, a reasonable explanation for equation (3) could be as follows: *"The rate of change of the recovered is proportional to the number of infected, where the proportionality constant is the recovery rate, γ. In other words, for a given unit of time, a proportion γ of the infected individuals move to the class of recovered individuals."*)

Question 2.

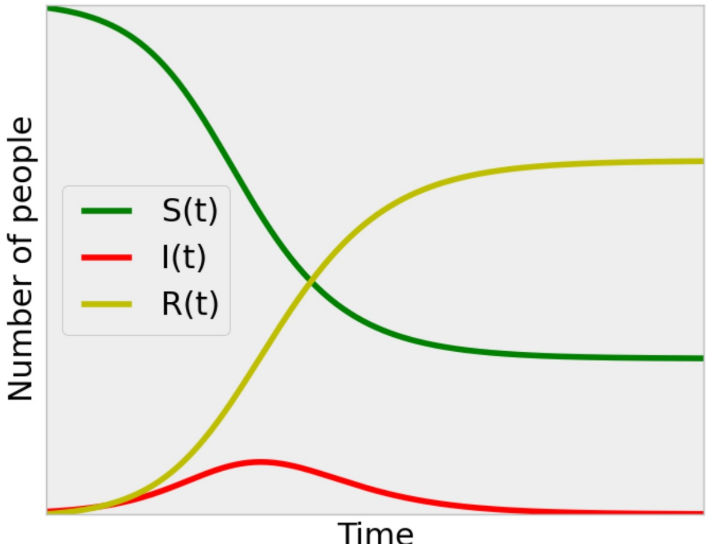
(a) Explain why equations (1)-(3) imply that

$$\frac{d}{dt}(S + I + R) = 0. \tag{4}$$

(b) Why does this make sense for the epidemic model?

Question 3. What are the equilibrium solutions of the SIR model? Based on this, what do we expect the number of infected individuals to be in the long run, if the situation has stabilized? Can we say anything about $\lim_{t\to\infty} S(t)$ and $\lim_{t\to\infty} R(t)$?

Question 4. Consider the graph below of $S(t)$, $I(t)$, and $R(t)$ as functions of time for given values of the parameters β and γ.



(a) Explain what is happening to the number of individuals in each of the three groups? Does the long term behavior of the number of infected individuals coincide with what you predicted based on the equilibrium solution?

(b) Make a prediction how the graph would change if we increased the infection rate β?

(c) Make a prediction how the graph would change if we increased the recovery rate γ?

(d) Consider the interactive graph given in the link: **Interactive Graph Link**. Does the graph support your prediction or not? Explain.

Question 5.

(a) Use the phase portrait (with Susceptibles on the horizontal axis and Infectives on the vertical axis) to describe how the equilibrium state changes as we decrease the rate of infection.

(b) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.3$, and the recovery rate $\gamma = 0.8$?

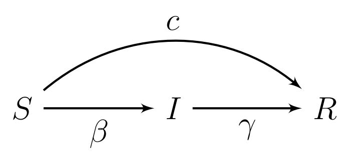
(c) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.4$, and the recovery rate $\gamma = 0.8$?

(d) How can you determine the equilibrium solution from the graphs of $S(t)$, $I(t)$, and $R(t)$ as functions of time (not from the phase portrait)?

SIR Epidemic Model with Vaccination

Now, assume we have the same model as above, but a vaccine was invented against the disease. The vaccine sends some of the Susceptibles directly to the Recovered (immune) state.

We can represent how individuals move from one category to another in the following way:



Question 6.

(a) Assume that a **fixed number**, c, of people are being vaccinated per time period. Modify the model given in equations (1)-(3) to account for the vaccination.

(b) Assume that a **fixed proportion**, c, of susceptible people are being vaccinated per time period. Modify the model given in equations (1)-(3) to account for the vaccination.

(c) Which of the two assumptions above makes more sense for a vaccination model? Explain your reasoning.

Question 7. Now, consider your model from Question 6(b). Find the equilibrium solutions in this case.
Hint: Note that when $c > 0$, $(-\beta I - c) \neq 0$.

Question 8.

(a) Remember that in Question 5 you found the equilibrium solution for the case when $I(0) = 0.5$, the infection rate $\beta = 0.3$, and the recovery rate $\gamma = 0.8$, with no vaccination. Now use the same parameters, but set the vaccination rate $c = 0.1$. Based on the figures, what does the equilibrium solution appear to be?

(b) Does this contradict your answer in Question 7?

(c) If there is a contradiction, give an explanation of why graphs may be misleading.

References

[1] James Murray, *Mathematical Biology: I. An Introduction*. Mathematical Biology, Springer, 2002.

[2] Cory Simon and Bernard Konrad, *The Mathematics Behind the Ebola Epidemic*.
http://corysimon.github.io/articles/the-mathematics-behind-the-ebola-epidemic/