Modeling Infectious Diseases

The SIR (Susceptible, Infected, Recovered) Epidemic Model

Team Members: 1. _____

2. _____

3. _____

4. _____

Objectives

To interpret and construct models using systems of ordinary differential equations in various settings.

Introduction

In this lab we will consider a classical epidemic model and interpret it in the setting of chickenpox. We will:

- Interpret the terms in a system of differential equations.
- Find equilibrium solutions and interpret them in the setting of the model.
- Study the behavior of equilibrium solutions as the parameters of the system vary. Equilibrium solutions are identified on the phase-portrait graph and on the component plot.
- Construct a new model that incorporates vaccination and analyze how vaccination changes the long-term behavior of solutions.

SIR Epidemic Model

Suppose we have a disease (like chickenpox) which, after recovery, provides immunity. In this model we will assume the number of individuals is constant, $N > 0$. Suppose that the disease is such that the population can be divided into three distinct classes:

1. The susceptibles, S, who can catch the disease;
2. The infected, I, who have the disease and can transmit it;
3. The recovered, R, who have had the disease and are recovered and immune to it.

Since the total number of individuals is constant,

$$S(t) + I(t) + R(t) = N.$$

where $S(t)$, $I(t)$, and $R(t)$ represent the number of individuals at time t in each class, respectively.

We can represent how individuals move from one category to another in the following way:

$$S \xrightarrow[\beta]{} I \xrightarrow[\gamma]{} R$$

where $\beta > 0$ is the **infection rate**, and $\gamma > 0$ is the **recovery rate**. We assume that the individuals in all classes are well-mixed, i.e., every pair of individuals has an equal probability of coming into contact with one another. In this case, our system can be described by a system of first order differential equations for the variables $S(t)$, $I(t)$, $R(t)$, given by

$$S' = -\beta S\, I \tag{1}$$
$$I' = \beta S\, I - \gamma I \tag{2}$$
$$R' = \gamma I \tag{3}$$

The physical interpretation of the different terms in each equation is the following.

- Equation (1) states that the number of susceptibles decreases at a rate proportional to the number of susceptibles and number of infected (the product is proportional to the number of encounters between susceptibles and infected).
- All the susceptibles who became infected move to the class of infected, therefore, in equation (2) we have that the number of infected increases at the same rate that the number of susceptibles decreases (the first term). The second term in equation (2) represents the number of infected decreasing because of the people who are recovering and moving into the class of recovered. This number is proportional to the number of infected, with a constant of proportionality, γ - the recovery rate.
- This equation says that the rate of change of the recovered, R', is proportional to the number of infected, I, where the proportionality constant is the recovery rate, γ. This equation means that for a given unit of time, a proportion γ of the infected individuals move to the class of recovered individuals.

Question 1. *(1 point)* Show that the equations (1)-(3) are consistent the hypothesis that the total number of individuals is constant in time.

Question 2. *(1 point)* Find the equilibrium solutions of the SIR model.

Question 3. *(3 points)* Consider the interactive graph given in the Math Studio link:

Interactive Graph

Turn on the graph for the solutions functions S, I and R by clicking on buttons **Suceptible-B**, **Infected-R**, and **Recovered-R**, respectively.

Also turn on the solution curve in the phase portrait by clicking on the **Sol-Curve-P**. In that phase portrait we have the Susceptible (S) on the horizontal axis and the Infected I on the vertical axis.

(3a) Set the system parameters to: Initial-Infected $I(0) = 0.1$, Infection-Rate $\beta = 0.5$, and Recovery-Rate $\gamma = 0.5$. Use the solution graphs and/or the solution curve for this case, to find the equilibrium solution of the system. What is the total population in this system?

(3b) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.3$, and the recovery rate $\gamma = 0.8$?

(3c) What is the equilibrium solution if $I(0) = 0.5$, the infection rate $\beta = 0.4$, and the recovery rate $\gamma = 0.8$?

SIR Epidemic Model with Vaccination

Now, assume we have the same model as above, but a vaccine was invented against the disease. The vaccine sends some of the susceptible directly to the Recovered (immune) state. We can represent how individuals move from one category to another in the following way:

$$\begin{tikzcd} S \arrow[r, "\beta"'] \arrow[rr, bend left, "c"] & I \arrow[r, "\gamma"'] & R \end{tikzcd}$$

Question 4. *(2 points)* Assume that a **fixed proportion**, c, of susceptible people are being vaccinated per time period. Modify the model given in equations (1)-(3) to account for the vaccination.

Question 5. *(2 points)* Find the equilibrium solutions for the SIR model with vaccination.

Hint: Since $c > 0$, then $(-\beta I - c) \neq 0$.

Question 6. *(1 point)* Set the system parameters to: Initial-Infected $I(0) = 0.5$, Infection-Rate $\beta = 0.3$, and Recovery-Rate $\gamma = 0.8$. Also set the vaccination rate $c = 0.2$. Use the solution graphs and/or the solution curve for this case, to find the equilibrium solution of the system. Compare this result with your answer to question **(3b)**.