# Water Tank Models

*We study simple mixing problems described by differential equations*

Team Members: 1. ____________________

2. ____________________

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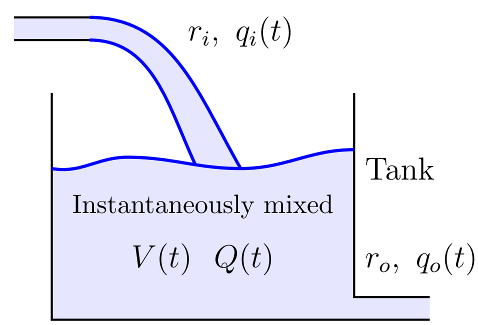
## Objectives

Students should be able to construct a mathematical model from basic physical concepts that describe the variable amount of salt in a water tank. Then, students should be able to solve the mathematical model and understand the physical predictions of the mathematical solution.

## Introduction

We focus on the system in the picture. A tank has an amount $Q(t)$ of salt dissolved in a volume $V(t)$ of water at a time $t$. Water is pouring into the tank at a rate $r_i(t)$ with a salt concentration $q_i(t)$. Water is also leaving the tank at a rate $r_o(t)$ with a salt concentration $q_o(t)$. A water rate $r$ means water volume per unit time while a salt concentration $q$ means amount of salt per unit volume.
We assume that the salt entering in the tank gets **instantaneously mixed**. As a consequence the salt concentration in the tank is **spatially homogeneous** at every time.




Before stating the problem we want to solve, we review the physical units of the main fields involved in it: $V$ has unit of volume; $Q$ has units of mass; both $r_i$ and $r_o$ have units of volume/time; and both $q_i$ and $q_o$ have units of mass/volume. The water tank problem above is described by three equations. The first equation is

$$V'(t) = r_i(t) - r_o(t).$$

This equation says that the rate of change in time of the volume of water in the tank is equal to the difference of volume time rates coming in and going out of the tank. The second equation is

$$Q'(t) = r_i(t)\, q_i(t) - r_o(t)\, q_o(t).$$

This equation is similar to the equation for the volume of water. This equation says that the rate of change in time of the amount of salt in the tank is equal to the difference of the time rates of the amount of salt coming in and going out of the tank. The product of a water rate, $r$, times a salt concentration, $q$, has units of amount of salt per time and it represents the amount of salt entering or leaving the tank per unit time.

The third equation is a consequence of mixing well the salt in the tank. When the salt in the tank is well mixed, the salt concentration is homogeneous in the tank, with value $Q(t)/V(t)$, that is,

$$q_o(t) = \frac{Q(t)}{V(t)}.$$

One final assumption, to simplify the calculations, is that both water rates, in and out, are constant in time,

$$r_i'(t) = r_o'(t) = 0.$$

**Question 1.** *(2 points)*

**(1a)** Find a differential equation for $Q(t)$, the amount of salt in the tank as a function of time. You do not need to solve this equation.

**(1b)** Describe what type of differential equation you found. For example, is it separable, Euler Homogeneous, linear, has constant coefficients, has variable coefficients, is homogenous, is non-homogeneous?

**Question 2.** At time $t = 0$ a tank contains 10 lb of salt dissolved in 100 gal of water. Water containing 1/4 lb of salt per gallon is constantly entering the tank at a rate of 3 gal/min. The well-stirred solution is constantly leaving the tank at the same rate.

**(2a)** *(1 point)* Describe what you think will happen over time with: the volume of water in the tank, and with the amount of salt in the tank. State your description with a verbal description or a plot.

**(2b)** *(1 point)* Write for this case the differential equation for $Q(t)$ you found in Question 1 and classify it. Also write the initial condition for this case.

**(2c)** *(1 point)* Find an expression for the amount of salt $Q(t)$ in the tank as function of time $t$.

**(2d)** *(1 point)* Describe the long term behavior of the amount of salt in the tank.

**Question 3.** Consider the problem in Question 2, with only one modification. The water rate leaving out of the tank is $r_o = 3.5$ gal/min.

**(3a)** *(1 point)* Describe what you think will happen over time with: the volume of water in the tank, and with the amount of salt in the tank. State your description with a verbal description or a plot.

**(3b)** *(1 point)* Write for this case the differential equation for $Q(t)$ you found in Question 1, and classify it. Also write the domain in time, $[0, t_{\max})$ when this equation holds, and indicate the reason for that.

**(3c)** *(1 point)* Find an expression for the amount of salt $Q(t)$ in the tank as function of time $t$.

**(3d)** *(1 point)* Find the salt concentration, $q_o(t) = Q(t)/V(t)$, of the last drop of water leaving the tank.