Forced Oscillators, Beating, and Resonance

Team Members: 1. ____________________

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Objectives

Understanding the concepts of beating and resonance in simple oscillating systems.

Introduction

In this lab we will study the oscillations in a mass-spring system that is subject to an external force. We focus on two cases:

- First we study the oscillations of this system in the case that the frequency of the driving force is close—but not equal—to the natural frequency of the string-mass system.
- Then we study what happens when the external driving force has a frequency exactly equal to the natural frequency of the system.

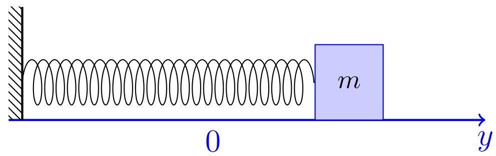
In the first case we will discover a particular type of behavior of the system, called beating. The oscillations of the system have a modulation in amplitude. This modulation has its own frequency, which depends on the difference between the driving and natural frequencies.

In the second case we discover the the behavior of the system called resonance. The amplitude of the oscillations grow without limit, and the physical system eventually breaks down. We also see how the beating solutions approximate the resonant solution.

The understanding of oscillations in simple systems—such as mass-spring systems—is the first step to understand vibrations in various engineering structures. The design of buildings and bridges must be such that resonance effects under the action of external forces—such as wind or earthquakes— are completely suppressed.

Forced Oscillations and Beating

A mass-spring system is a mass m attached to a spring with spring constant k that slides on a frictionless table. We denote by $y(t)$ the displacement of the mass as a function of time, where $y = 0$ represents the rest position of the mass.



This system is described by Newton's law of motion $ma = f$, where $a = y''$, and f is given by Hooke's law, $f(y) = -k\,y$. So the equation of motion for the oscillating spring without any external force is

$$m\,y'' + k\,y = 0.$$

The **natural frequency** of the oscillator is the frequency of all oscillations when there is no external force acting on the system.

Question 1. (No external force) Write the general solution for the mass-spring system in the case that there are no external forces, and find a formula for the **natural frequency** of the system in terms of m and k.

Note. (Non-resonant force) Consider now a spring-mass system with mass $m = 1$ grams, and $k = 25$ grams per second square. We now add an external force to the mass-spring system given by

$$f_{\rm ext}(t) = \cos(\nu t), \qquad \nu > 0.$$

The frequency of the external force, ν, is called the **driving frequency**.

Recall from lecture that the solution to the initial value problem

$$y'' + 25\,y = \cos(\nu t), \quad \nu \neq 5, \quad \text{and} \quad y(0) = 0, \quad y'(0) = 0.$$

is given by

$$\boxed{y(t) = \frac{1}{(25 - \nu^2)}\big(\cos(\nu t) - \cos(5t)\big)}. \tag{1}$$

Question 2. Look at the solution for an oscillator with non-resonant force, given by formula (1). Based on the formula, what happens to the amplitude of the oscillations as the driving frequency, ν, approaches the natural frequency, $\omega = 5$? Explain your reasoning.

Resonant Oscillator

The oscillator is in **resonance** when the driving frequency ν of the external force is equal to the natural frequency $\omega = 5$ of the unperturbed oscillator. The equation that describes a resonant oscillator is

$$y'' + 25\,y = \cos(5t),$$

Recall from lecture that the resonant initial value problem

$$y'' + 25\,y = \cos(5t), \qquad \text{and} \quad y(0) = 0, \quad y'(0) = 0,$$

has the solution

$$\boxed{y(t) = \frac{1}{10} t \sin(5t)}. \tag{2}$$

Question 3. Look at the solution for an oscillator with resonant force, given by formula (2). Based on the formula, what happens to the amplitude of the oscillations as time, t, goes by? Explain your reasoning.

Graphical Analysis of the Resonant and Non Resonant Solutions

Consider the interactive graph given in the link: **Interactive Graph Link**. The functions displayed in this graph are:

$$y_{\text{Resonant}}(t) = \frac{t}{10}\,\sin(5t), \qquad y_{\text{NonResonant}}(t) = \frac{1}{(25-\nu^2)}\big(\cos(\nu t) - \cos(5t)\big) \tag{3}$$

$$y_{p\text{NonResonant}}(t) = \frac{1}{(25-\nu^2)}\cos(\nu t), \qquad y_{hom\text{NonResonant}}(t) = -\frac{1}{(25-\nu^2)}\cos(5t) \tag{4}$$

We try to understand how the non-resonant solution $y_{\text{NonResonant}}$ approaches the resonant solution y_{Resonant} as the driving frequency ν approaches the natural frequency 5, that is,

$$y_{\text{NonResonant}} \quad \underset{\nu\to 5}{\longrightarrow} \quad y_{\text{Resonant}}$$

Question 3.

(a) An oscillatory function, called carrier signal, has **beating** when the carrier signal has a periodic modulation in amplitude with frequency smaller than the carrier signal frequency. Find the minimum value of the driving frequency ν such that the solution $y_{\text{NonResonant}}$ displays beating.

Hint: In the Interactive Graph, turn on $y_{\text{NonResonant}}$, turn off all the other functions, and vary the driving frequency ν from 0 to 5.

(b) Based on the graph, describe in your own words the beating phenomenon.

(c) Discuss with your partners what beating sounds like, if the solution of the ODE describes sound waves.

(d) Watch the following short video: https://www.youtube.com/watch?v=V8W4Djz6jnY. What are the frequencies of the two tuning forks? Explain why beating occurs.

(e) For $\nu = 4.7$, describe how are the graphs of $y_{p\text{NonResonant}}$ and $y_{hom\text{NonResonant}}$ related when the beating solution $y_{\text{NonResonant}}$ has high amplitude, and when the beating solution $y_{\text{NonResonant}}$ has low amplitude. Is this behavior consistent with the expressions in (3)-(4)? Why?.

(f) Describe how the non-resonant solution $y_{\text{NonResonant}}$ approaches the resonant solution y_{Resonant} as the driving frequency ν approaches the natural frequency 5.

(g) What is the behavior of the amplitude of y_{Resonant} as the time variable $t \to \infty$? What will happen to the actual spring of a resonant system when time is large enough?