Section 7.2 : Overview of Fourier Series

Plan:
* Fourier Expansion of Functions
* Odd and Even Functions
* Sine and Cosine Series

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* The Fourier Expansion of Functions

Thrm: The set in V

$$F = \left\{ u_0 = \frac{1}{2},\quad u_n(x) = \cos\left(\frac{n\pi x}{L}\right),\quad v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

is an orthogonal set.

Proof: show that:

$$u_n \cdot u_m = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{for } n \neq m \\ L & \text{for } n = m \neq 0 \\ 2L & \text{for } n = m = 0 \end{cases}$$

use: $\cos(\theta)\cos(\phi) = \frac{1}{2}\left(\cos(\theta+\phi) + \cos(\theta-\phi)\right)$.

$$v_n \cdot v_m = \int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & \text{for } n \neq m \\ L & \text{for } n = m \neq 0 \end{cases}$$

use: $\sin(\theta)\sin(\phi) = \frac{1}{2}\left(\cos(\theta-\phi) - \cos(\theta+\phi)\right)$

$$u_n \cdot v_n = \int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = 0 \qquad \text{all } n, m.$$

use: $\sin(\theta)\cos(\phi) = \frac{1}{2}\left(\sin(\theta+\phi) + \sin(\theta-\phi)\right)$.

Remark: $u_0 \cdot u_0 = \frac{1}{4} 2L \Rightarrow \|u_0\| = \sqrt{\frac{L}{2}}$

$u_n \cdot u_n = L \Rightarrow \|u_n\| = \sqrt{L}$ $\quad n \neq 0$

$V_n \cdot V_n = L \Rightarrow \|V_n\| = \sqrt{L}$

An orthonormal set (like $\{i, j, k\}$) is

$$\left\{ \tilde{u}_0 = \frac{1}{\sqrt{2L}}, \quad \tilde{u}_n(x) = \frac{1}{\sqrt{L}} \cos\left(\frac{n\pi x}{L}\right), \quad \tilde{V}_n(x) = \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi x}{L}\right) \right\}_{n=1}^{\infty}$$

People uses the orthogonal set.

The Fourier Expansion Theorem

Thrm: The orthogonal set F is a basis for the subspace $W \subset V$ of continuous functions on $[-L, L]$ and for every $f \in W$ the function

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

Satisfies $f_F(x) = f(x)$, where

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx \qquad \left(= \frac{2}{L} f \cdot u_0 \right)$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad \left(= \frac{1}{L} f \cdot u_n \right)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad \left(= \frac{1}{L} f \cdot v_n \right)$$

Thrm: Furthermore, if f is piecewise continuous, the function f_F satisfies

$$f_F(x) = f(x),$$

where f is continuous, and

$$f_F(x_0) = \frac{1}{2}\left(\lim_{x\to x_0^+} f(x) + \lim_{x\to x_0^-} f(x)\right)$$

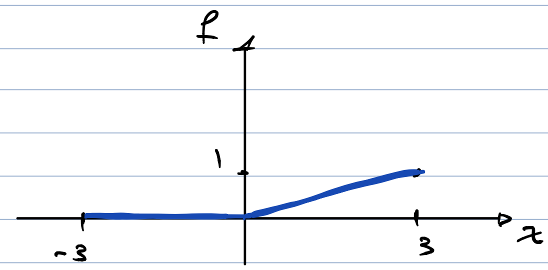
for all x_0 where f is discontinuous.

Remark: * Instead of infinite sums, we can use finite sums approximations:

$$f_{FN}(x) = \frac{a_0}{2} + \sum_{n=1}^{N}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

Example: Find f_F for $f(x) = \begin{cases} \frac{x}{3} & x \in [0,3] \\ 0 & x \in [-3,0] \end{cases}$

Sol:



$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad ; \quad L = 3$$

$$= \frac{1}{3}\int_{-3}^{3} f(x) \sin\left(\frac{n\pi x}{3}\right) dx$$

$$= \frac{1}{3}\left(\underbrace{\int_{-3}^{0} 0 \sin\left(\frac{n\pi x}{3}\right) dx}_{=0} + \int_{0}^{3} \frac{x}{3} \sin\left(\frac{n\pi x}{3}\right) dx\right)$$

$$b_n = \frac{1}{9}\int_{0}^{3} x \sin\left(\frac{n\pi x}{3}\right) dx \qquad \text{(Integration Table)}$$

$$= \frac{1}{9}\left(-\frac{3}{n\pi} x \cos\left(\frac{n\pi x}{3}\right) + \frac{9}{n^2\pi^2} \sin\left(\frac{n\pi x}{3}\right)\right)\Big|_0^3$$

$$= \frac{1}{9}\left(-\frac{9}{m\pi}\cos(n\pi) + 0 - (0 + 0)\right)$$

$$b_n = -\frac{1}{n\pi}\cos(n\pi) \quad , \quad \cos(n\pi) = (-1)^n$$

$$\boxed{b_n = \frac{(-1)^{n+1}}{n\pi}}$$

Analogously $\Rightarrow$ $\boxed{a_n = \frac{((-1)^n - 1)}{n^2\pi^2}}$

$\Rightarrow$ $\boxed{a_0 = \frac{1}{2}}$

$$\boxed{f_F(x) = \frac{1}{4} + \sum_{n=0}^{\infty}\left(\frac{((-1)^n - 1)}{n^2\pi^2}\cos\left(\frac{n\pi x}{3}\right) + \frac{(-1)^{n+1}}{n\pi}\sin\left(\frac{n\pi x}{3}\right)\right)}$$

* Even and Odd Functions

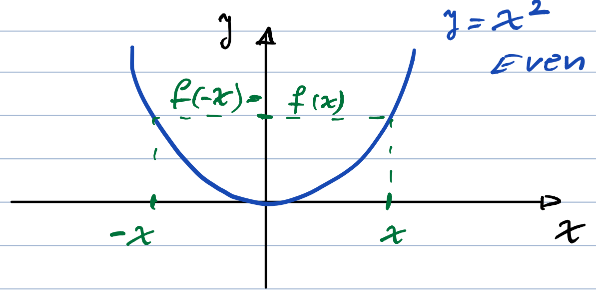
Def: A function f on $[-L, L]$ is:

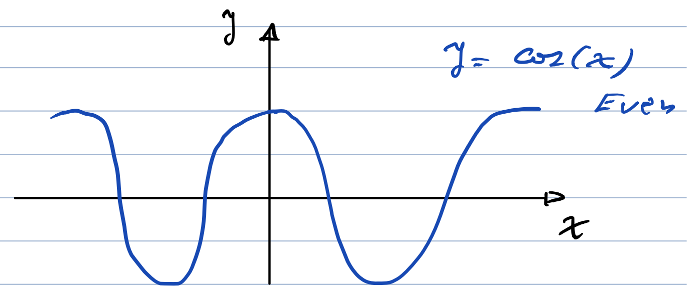
Even iff $f(-x) = f(x)$ for all $x \in [-L, L]$.

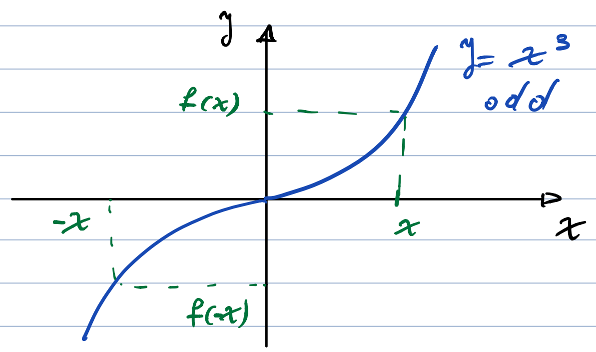
Odd iff $\underbrace{f(-x) = -f(x)}$ for all $x \in [-L, L]$.

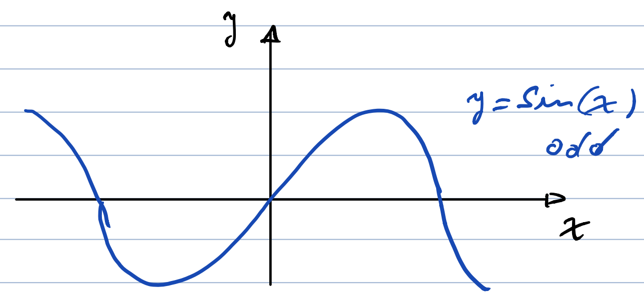
($f(0) = -f(0) \Rightarrow \underline{f(0) = 0 \quad f \text{ odd}}$)

Example:



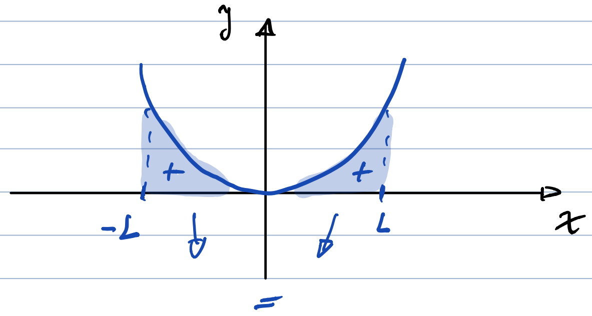


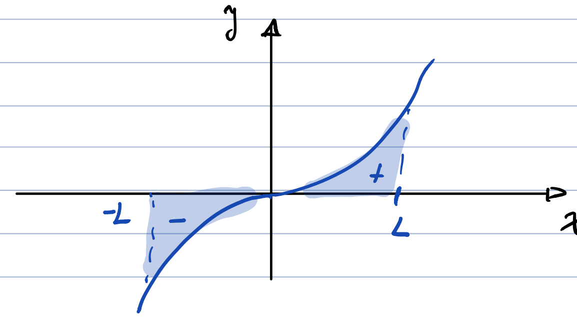




Properties: If f_e, g_e even, h_o, l_o odd, then

* $a f_e + b g_e$ even
* $a h_o + b l_o$ odd
* $f_e g_e$ even
* $h_o l_o$ $\underline{\text{even}}$
* $f_e h_o$ $\underline{\text{odd}}$
* $\displaystyle\int_{-L}^{L} h_o(x)\, dx = 0$
* $\displaystyle\int_{-L}^{L} f_e(x)\, dx = 2 \int_{0}^{L} f_e(x)\, dx$





Thrm: Given f on $[-L, L]$ with

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right) \quad (1)$$

(a) If f is even, then $b_n = 0$ and (1) is called a cosine series,

$$f_F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right).$$

(b) If f is odd, then $a_n = 0$ and (1) is called a cosine series,

$$f_F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right).$$

Proof:

(a) $$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{even}} \; \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\underline{\text{odd}}} \, dx = 0.$$

(b) $$a_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \; \underbrace{\cos\left(\frac{n\pi x}{L}\right)}_{\text{even}}}_{\underline{\text{odd}}} \, dx = 0$$

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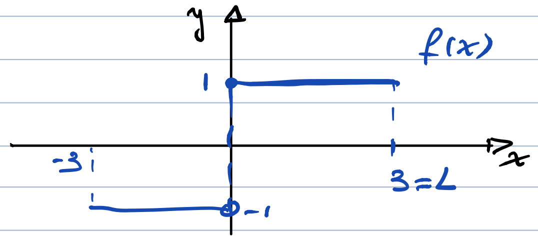
Example: Find f_F for $f(x) = \begin{cases} -1 & x \in [-3, 0) \\ +1 & x \in [0, 3] \end{cases}$

Sol:



$f(x)$ odd $x \in [-3, 3] - \{0\}$

$$f_F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{1}{L} \int_{-L}^{L} \underbrace{\underbrace{f(x)}_{\text{odd}} \; \underbrace{\sin\left(\frac{n\pi x}{L}\right)}_{\text{odd}}}_{\text{even}} \, dx = \frac{2}{L} \int_0^L \underbrace{f(x)}_{=1} \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{3} \int_0^3 \sin\left(n\pi \frac{x}{3}\right) dx$$

$$= \frac{2}{3}\left(-\frac{3}{n\pi}\right) \cos\left(n\pi \frac{x}{3}\right) \Big|_0^3$$

$$= -\frac{2}{n\pi}\left(\cos(n\pi) - 1\right) \quad ; \quad \cos(n\pi) = (-1)^n$$

$$\boxed{b_n = \frac{2}{n\pi}\left(1 - (-1)^n\right)}$$

$$\boxed{f_F(x) = \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n\pi} \sin\left(n\pi \frac{x}{3}\right)}$$

odd.

$f_F(0) = 0$