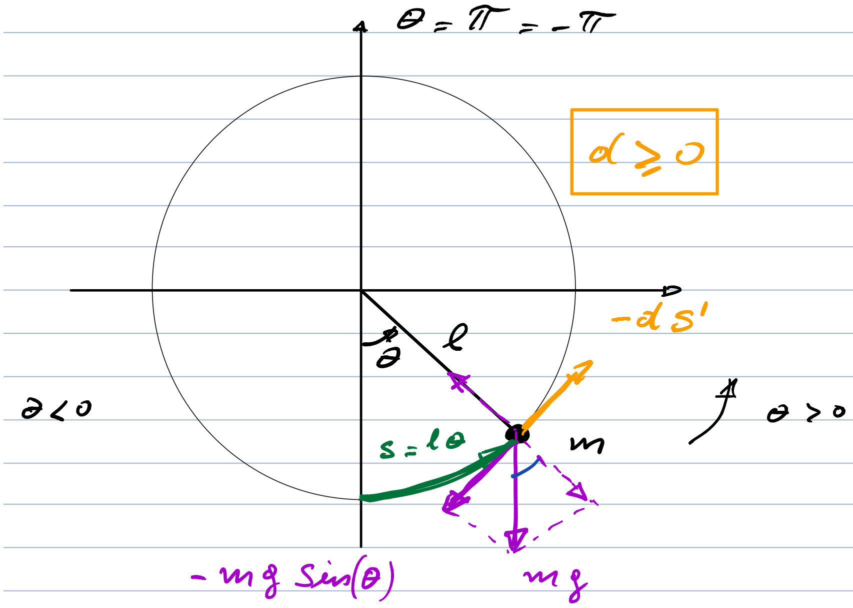
* The Nonlinear Pendulum

Example: [Write the Nonlinear Pendulum equations as a first order system]



$$m a = f$$

$$a = s'' = (\ell\,\theta(t))'' = \ell\,\theta''$$

$$f_m = -mg\,\sin(\theta)$$

$$f_\alpha = -\alpha s' = -\alpha\,\ell\theta'$$

$$m\ell\theta'' = -mg\,\sin(\theta) - \alpha\,\ell\,\theta'$$

$$\boxed{\theta'' = -\frac{g}{\ell}\sin(\theta) - \frac{\alpha}{m}\theta'}$$

second order, Nonlinear

First Order Reduction:

$$\left.\begin{aligned} u &= \theta \\ v &= \theta' \end{aligned}\right\} \Rightarrow$$

$$u' = v$$

$$v' = \theta'' = -\frac{g}{\ell}\sin(\theta) - \frac{\alpha}{m}\theta'$$

$$= -\frac{g}{\ell}\sin(u) - \frac{\alpha}{m}v$$

$$\boxed{\begin{aligned} u' &= v \\ v' &= -\frac{g}{\ell}\sin(u) - \frac{\alpha}{m}v \end{aligned}}$$

F.O.R.
Nonlinear

$$\vec{x} = \begin{bmatrix} u \\ v \end{bmatrix}, \qquad \vec{f} = \begin{bmatrix} v \\ -\frac{g}{l}\sin(u) - \frac{d}{m} v \end{bmatrix} \Rightarrow \boxed{\vec{x}' = \vec{f}(\vec{x})}$$

$\blacksquare$

<u>Remark</u>: The Small Oscillations Approximation is

$$|\theta| \ll 1 \Rightarrow \boxed{\sin(\theta) \sim \theta}$$

The result is a linear system:

$$\boxed{\begin{aligned} u' &= v \\ v' &= -\frac{g}{l} u - \frac{d}{m} v \end{aligned}}$$

$$\boxed{\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}}$$

Mathematically equivalent to a Spring.

* <u>Equilibrium Solutions</u>

<u>Remark</u>: For simplicity: Choose $\boxed{\frac{g}{l} = 1}$, $\boxed{m = 1}$.

$$\begin{aligned} u' &= v \\ v' &= -\sin(u) - d\, v \end{aligned} \qquad (1)$$

<u>Example</u>: [Find the critical Points of (1).]

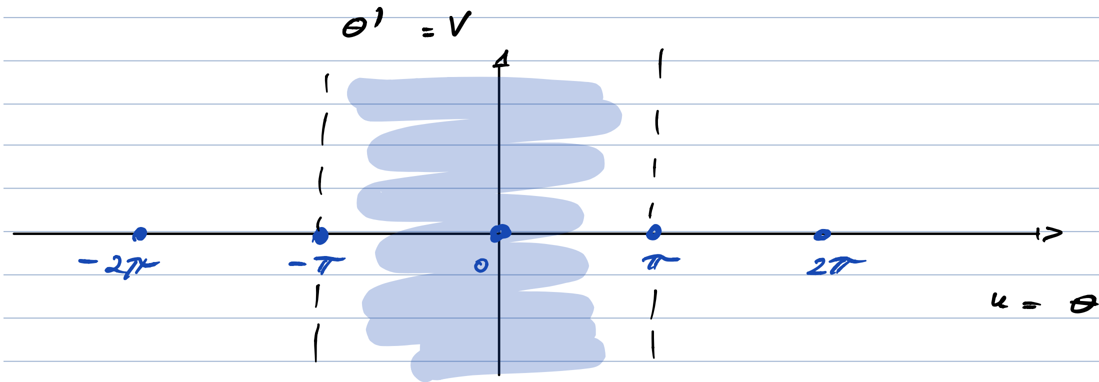
<u>Sol</u>.

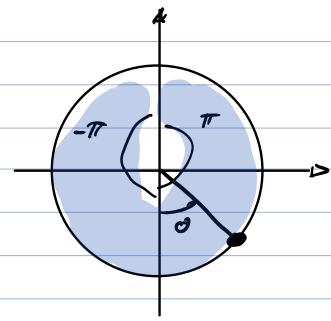
$$\vec{f}(\vec{x}) = \begin{bmatrix} v \\ -\sin(u) - d\, v \end{bmatrix}, \qquad \vec{f} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{array}{l} \boxed{v = 0} \\ -\sin(u) - \alpha v = 0 \end{array}\right\} \Rightarrow \quad \sin(u) = 0$$

$$\boxed{u_n = n\pi} \qquad n : \text{integer}$$

$$\boxed{\vec{x}_{c_n} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}} \Rightarrow \boxed{P_{c_n} = (n\pi, 0)} \qquad n = 0, \pm 1, \pm 2, \dots$$





Example: Find the linearizations of (L) at the critical points

Sol:

$$\vec{f}(\vec{x}) = \begin{bmatrix} v \\ -\sin(u) - \alpha v \end{bmatrix}$$

$$Df(\vec{x}) = \begin{bmatrix} \partial_u v & \partial_v v \\ \partial_u(-\sin(u) - \alpha v) & \partial_v(-\sin(u) - \alpha v) \end{bmatrix}$$

$$\boxed{Df(\vec{x}) = \begin{bmatrix} 0 & 1 \\ -\cos(u) & -\alpha \end{bmatrix}}$$

Derivative Matrix

$$\boxed{\alpha \geq 0}$$

Now we need to evaluate $Df(\vec{x})$ at the equilibrium sols. $\vec{x}_{c_n} = \begin{bmatrix} n\pi \\ 0 \end{bmatrix}$

Equivalently, at the critical points

$P_{c_n} = (n\pi, 0)$. Since $\cos(n\pi) = (-1)^n$

we have two types of critical points:

Even: $E_{c_k} = 2k\pi$ $\quad k = 0, \pm 1, \pm 2, \ldots$

Odd: $O_{c_k} = (2k+1)\pi$

Therefore, the linearizations are:

$$\boxed{Df(E_{c_k}) = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}} \quad \text{and} \quad \boxed{Df(O_{c_k}) = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix}}$$

$k = 0, \pm 1, \pm 2, \ldots$

<u>Recall</u>: The sols of $\vec{x}' = \vec{f}(x)$ (pendulum)

are close to solutions of

$$\vec{u}' = Df_c \, \vec{u}$$

near the equilibrium sols $\vec{x}_c$.