Section 6.4 : Applications of Qualitative Analysis

Plan:
- * Competing Species: ✓
- * Predator-Prey ✓
- * The Non-Linear Pendulum ←

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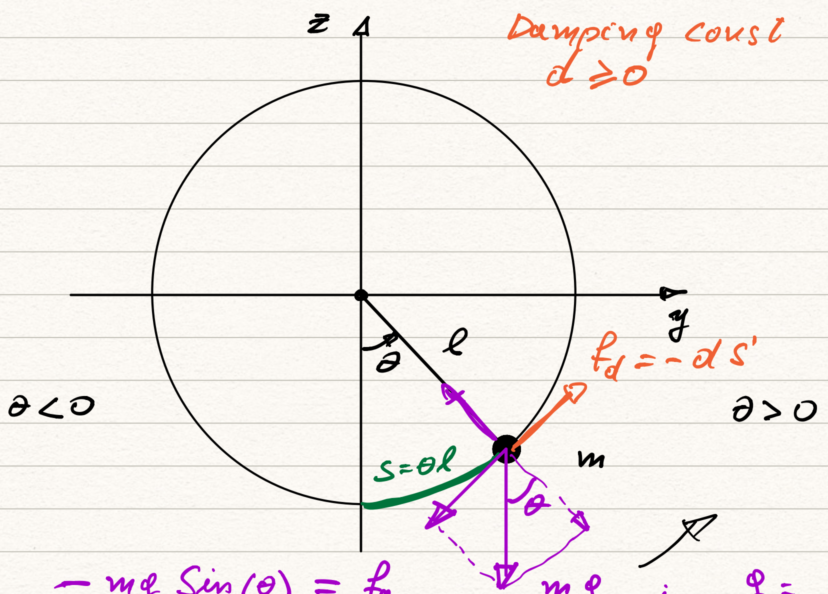
* The Non-Linear Pendulum

$\ldots = -3\pi = -\pi = \theta = \pi = 3\pi = \ldots$



; $g = 9.81 \; \frac{m}{s^2}$

$\theta = 0$

$\ldots = -2\pi = \theta = 2\pi = \ldots$

Newton's Eq.

$m a = f$

$a = s'' = (\ell\theta)'' = \ell\theta''$

ℓ : const , $\theta(t)$

$f = f_g + f_d$

$f_g = -mg\sin(\theta)$

$f_d = -\alpha s' = -\alpha \ell \theta'$

$$\boxed{m\ell\theta'' = -mg\sin(\theta) - \alpha\ell\theta'}$$

$$\boxed{\theta'' = -\frac{g}{\ell}\sin(\theta) - \frac{\alpha}{m}\theta'}$$

SO NL DE $\quad \boxed{\theta(t)}$

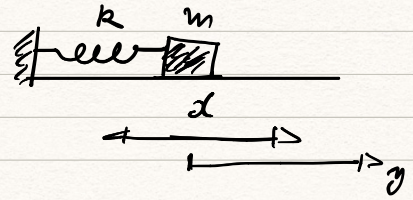
Remark: Small oscillations

$|\theta| \ll 1 \Rightarrow \sin(\theta) \simeq \theta$ (Taylor expans.)

$$\theta'' \simeq -\frac{g}{l}\theta - \frac{\alpha}{m}\theta'$$

$$\boxed{m\theta'' + \alpha\theta' + \frac{gm}{l}\theta = 0}$$

Spring:



Mathematically equivalent.

$$\boxed{my'' + \alpha y' + ky = 0}$$

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Sketch phase portrait of sols. N.P.

$$\boxed{\theta'' = -\frac{g}{l}\sin(\theta) - \frac{\alpha}{m}\theta'} \quad ; \quad \theta(t) \qquad (1)$$

Ex: [write (1) as a first order system.]

Sol

$x_1(t) = \theta(t)$ angular position

$x_2(t) = \theta'(t)$ angular velocity.

$\Rightarrow \underline{x_1' = x_2}$

$$x_2' = \theta'' = -\frac{g}{l}\sin(\theta) - \frac{\alpha}{m}\theta'$$

$$= -\frac{g}{l}\sin(x_1) - \frac{\alpha}{m}x_2$$

$$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -\frac{g}{l}\sin(x_1) - \frac{\alpha}{m}x_2 \end{aligned}}$$

SFO NL DE.

■

Example: Sketch a phase portrait of sols of

$$\begin{bmatrix} x_1' = x_2 \\ x_2' = -\sin(x_1) - \alpha x_2 \end{bmatrix}$$

Remark: For simplicity we chose: $\boxed{\dfrac{g}{\ell} = 1}$, $\boxed{m = 1}$.

Sol.

Vector Field: $\vec{F}(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \alpha x_2 \end{bmatrix}$

critical points. $(\vec{F} = \vec{0})$

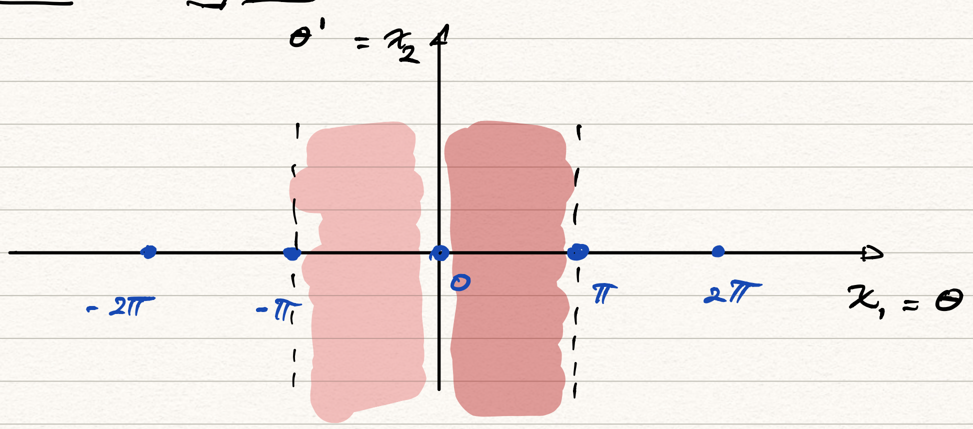
$$\left.\begin{aligned} x_2 &= 0 \quad \checkmark \\ -\sin(x_1) - \alpha x_2 &= 0 \end{aligned}\right\} \Rightarrow \quad \underline{\sin(x_1) = 0}$$

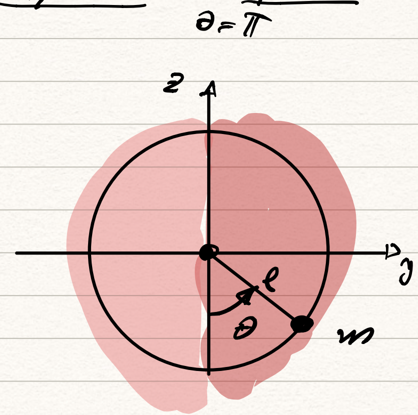
$x_1 = n\pi$

$\underline{n = 0, \pm 1, \pm 2, \dots}$

$$\boxed{x^n = (n\pi, 0)}$$

Phase Space **Physical Space**





Derivative Matrix and Linearizations

$$\vec{F}(x) = \begin{bmatrix} x_2 \\ -\sin(x_1) - \alpha x_2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \quad DF(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix}$$

$$DF(x) = \begin{bmatrix} 0 & 1 \\ -\cos(x_1) & -d \end{bmatrix} \qquad \underline{d \geq 0}$$

$$x^n = (n\pi, 0) \Rightarrow DF(x^n) = \begin{bmatrix} 0 & 1 \\ -\cos(n\pi) & -d \end{bmatrix} ; \cos(n\pi) = (-1)^n$$

$$DF(x^n) = \begin{bmatrix} 0 & 1 \\ -(-1)^n & -d \end{bmatrix} \Rightarrow \boxed{DF(x^n) = \begin{bmatrix} 0 & 1 \\ (-1)^{n+1} & -d \end{bmatrix}}$$

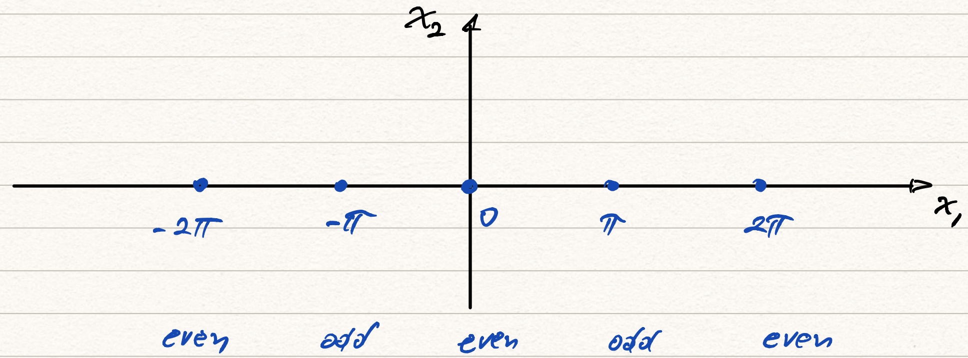
<u>n even</u> <u>$n = 2k$</u> ; <u>n odd</u> <u>$n = 2k+1$</u>

$$(-1)^{n+1} = (-1)^{2k+1} = (-1)^{2k}(-1) = \left((-1)^2\right)^k(-1) = 1^k(-1) = -1$$

$$(-1)^{n+1} = (-1)^{2k+1+1} = (-1)^{2k+2} = (-1)^{2(k+1)} = \left((-1)^2\right)^{k+1} = 1$$

$$\boxed{DF(x^{2k}) = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}} \qquad \boxed{DF(x^{2k+1}) = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix}}$$

<u>even c.p.</u> <u>odd c.p.</u>



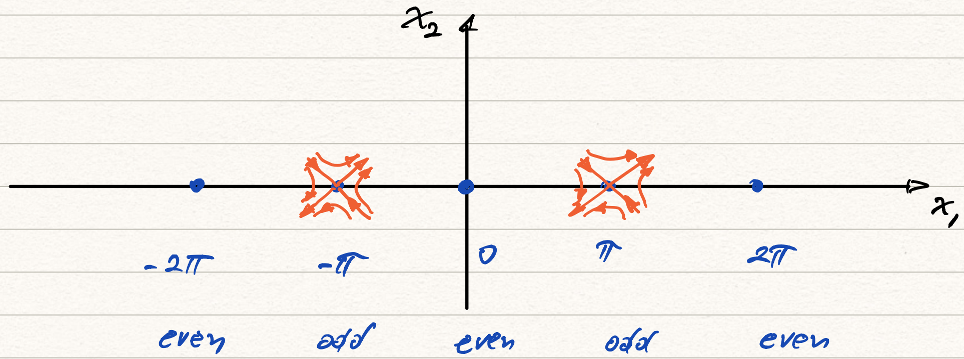
* Linearizations : No Friction $\boxed{d = 0}$

odd critical points : $x^{2k+1} = ((2k+1)\pi, 0)$

$DF(x^{2k+1}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: Eigenpairs: (some work)

$\lambda_\pm = \pm 1$, $\vec{v}_\pm = \begin{bmatrix} \pm 1 \\ 1 \end{bmatrix}$

$\lambda_- < 0 < \lambda_+$: x^{2k+1} Saddle Nodes



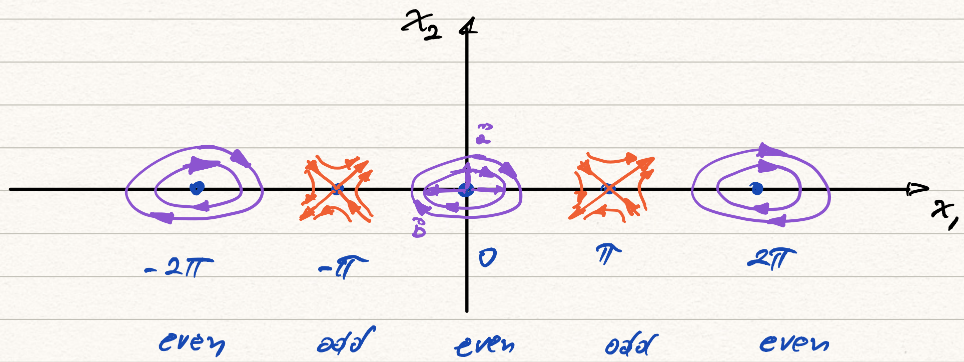
Even critical points : $x^{2k} = (2k\pi, 0)$

$DF(x^{2k}) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; Eigenpairs : (some work)

$\boxed{\lambda_\pm = \pm i}$ $\vec{v}_\mp = \begin{bmatrix} \mp i \\ 1 \end{bmatrix}$

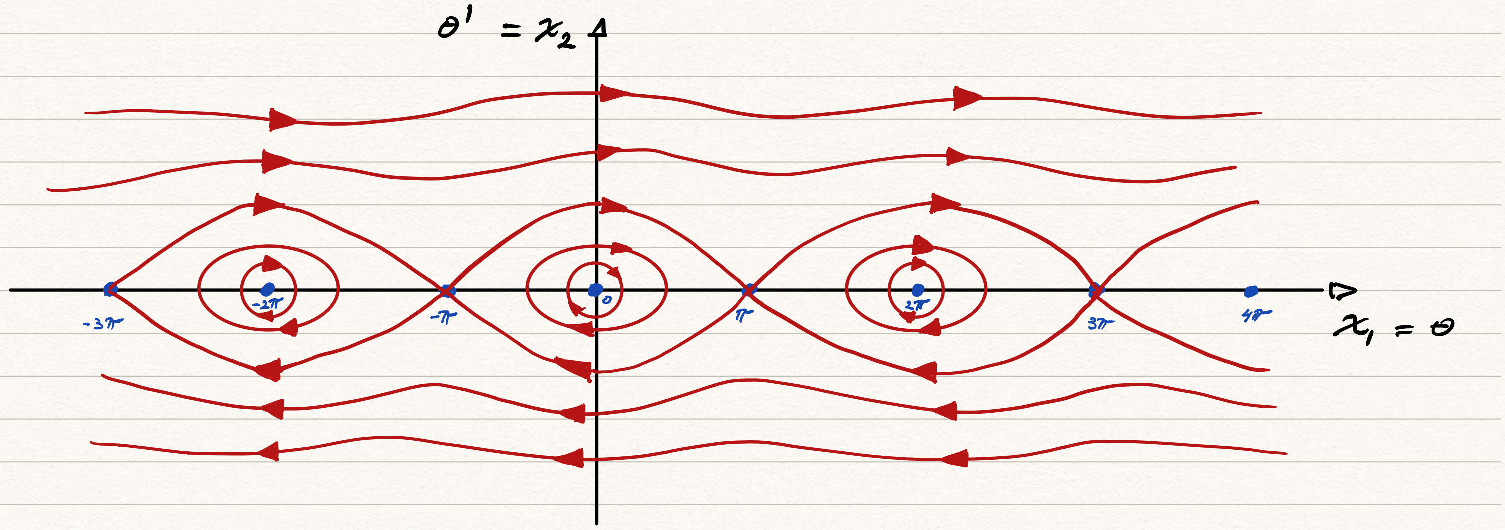
$x^{2k} = (2k\pi, 0)$ Centers ← No Hartman-Grobman

Further Study (Direction Fields)



$$\vec{V}_+ = \begin{bmatrix} -i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 + (-1)\,i \\ 1 + 0\,i \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i \quad , \quad \vec{a} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$\vec{a} \to -\vec{b}$$



$\underline{d = 0}$ No Friction

* Linearizations $d > 0$ (with Friction)

odd critical points: $\quad x^{2k+1} = \left((2k+1)\pi, 0 \right)$

$$DF(x^{2k+1}) = \begin{bmatrix} 0 & 1 \\ 1 & -d \end{bmatrix}$$

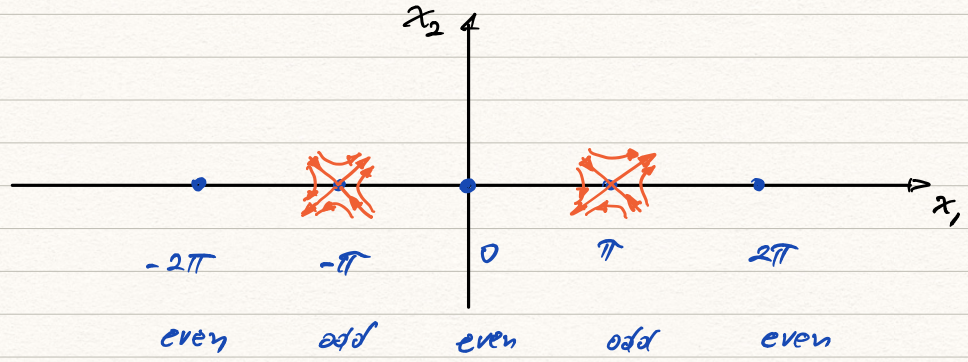
Eigenpairs
(some work)

$$\lambda_\pm = \frac{1}{2}\left(-d \pm \sqrt{d^2+4}\right) \quad ; \quad \vec{V}_\pm = \begin{bmatrix} d \pm \sqrt{d^2+4} \\ 2 \end{bmatrix}$$

$$\lambda_- = -\frac{d}{2} - \frac{1}{2}\sqrt{d^2+4} < 0$$

$$\lambda_+ = -\frac{d}{2} + \frac{1}{2}\sqrt{d^2+4} > 0$$

$\underline{x^{2k+1} \quad \text{Saddle Nodes}}$



<u>Even critical points</u> $\quad x^{2k} = (2k\pi, 0)$

$$DF(x^{2k}) = \begin{bmatrix} 0 & 1 \\ -1 & -d \end{bmatrix}$$

Eigenpairs : $\quad \lambda_\pm = \frac{1}{2}\left(-d \pm \sqrt{d^2-4}\right) \qquad \vec{v}_\pm = \begin{bmatrix} -d \mp \sqrt{d^2-4} \\ 2 \end{bmatrix}$

(some work)

$\boxed{d > 0}$

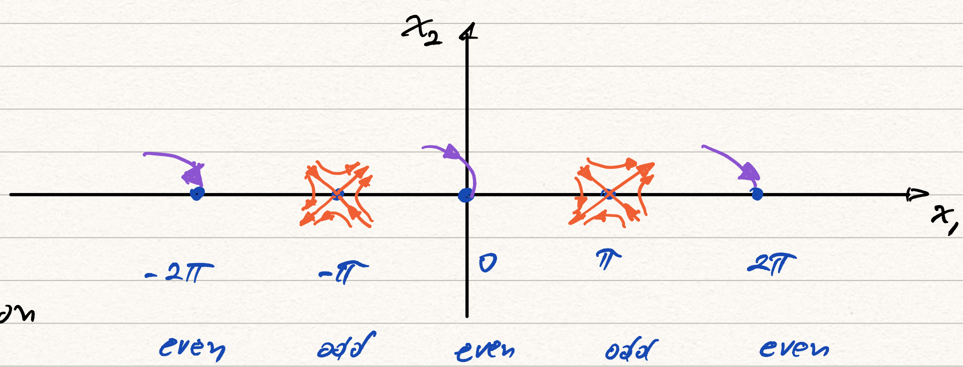
Two cases: $\begin{cases} (A) & d^2-4 \geq 0 \implies d \geq 2 \\ (B) & d^2-4 < 0 \implies 0 < d < 2 \end{cases}$

<u>Case (A)</u> : $d \geq 2$, large friction $\implies d^2-4 \geq 0$

$\lambda_\pm = -\frac{d}{2} \pm \frac{1}{2}\sqrt{d^2-4}$ $\quad$ <u>Real</u>.

$\lambda_- < 0$; $\quad \lambda_+ = -\frac{d}{2} + \frac{1}{2}\underbrace{\sqrt{d^2-4}}_{< d} < 0$

x^{2k} $\quad$ <u>Sink Nodes</u> $\quad$ (No oscillations)



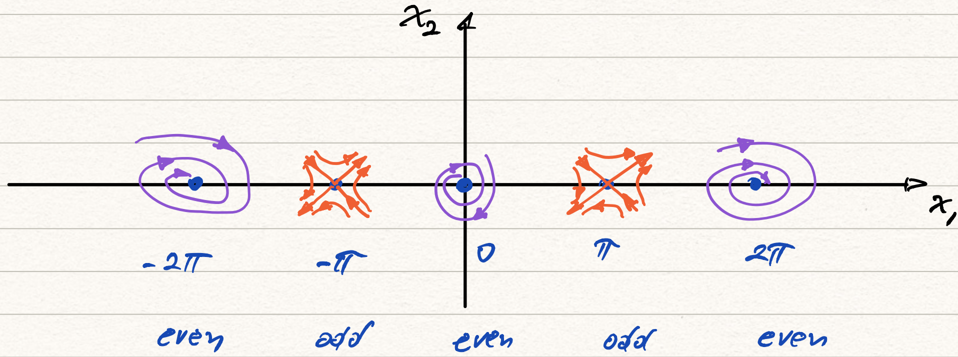
$d \geq 2$
large friction

Case (B) $\quad 0 < d < 2$, Small friction , $\Rightarrow$ $d^2 - 4 < 0$

$$\lambda_\pm = -\frac{d}{2} \pm \frac{1}{2}\sqrt{\underbrace{d^2-4}_{\text{negative}}} = \underbrace{-\frac{d}{2}}_{\substack{\Downarrow \\ \text{Real Part} \\ \text{negative}}} \pm \frac{1}{2}\underbrace{\sqrt{4-d^2}}_{\text{Positive}}\, i$$

complex

x^{2k} Sink Spirals (oscillatory with decay



$0 < d < 2$, Small friction

x^{2k+1} Saddle Nodes

x^{2k} : $\begin{cases} d = 0 & \text{centers (oscillations without decay)} \\ 0 < d < 2 & \text{Sink Spirals (oscillations with decay)} \\ d \geq 2 & \text{Sink Nodes (decay without oscillations)} \end{cases}$

case: Small friction, $0 < d < 2$.