Section 6.4: Applications of Qualitative Analysis

Plan:
* Competing Species:
* Predator-Prey
* The Non-Linear Pendulum

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* **Review**: Sketching Phase Portraits (diagrams)

Autonomous system: $\vec{x}' = \vec{F}(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$

Find Critical Points: $\vec{F}(x^c) = \vec{0}$.
(equilibrium Solutions)

Find Derivative Matrix : $DF(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}$
(Jacobian)

Use eigenpairs of $DF(x^c)$ to characterize each critical point.

Use all above information to sketch the phase portrait.

* Competing Species : (Finite Food)

Example : (Extinction)

Sketch a phase portrait of sols of

$$r' = 3r\left(1 - \frac{r}{3}\right) - 2rs$$

$$s' = 2s\left(1 - \frac{s}{2}\right) - rs$$

Sol

r : rabbits $\quad r_r = 3 \quad K_r = 3 \quad \alpha = -2$

s : sheep $\quad r_s = 2 \quad K_s = 2 \quad \beta = -1$

$$\vec{F}(x) = \begin{bmatrix} 3r\left(1-\frac{r}{3}\right) - 2rs \\ 2s\left(1-\frac{s}{2}\right) - rs \end{bmatrix}$$

Critical Points : $\quad \vec{F}(x^c) = \vec{0}.$

$$\left.\begin{aligned} 3r\left(1-\frac{r}{3}\right) - 2rs &= 0 \\ 2s\left(1-\frac{s}{2}\right) - rs &= 0 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} r\,(3 - r - 2s) &= 0 \\ s\,(2 - s - r) &= 0 \end{aligned}\right\} \Rightarrow \quad r=0,\ s=0 \Rightarrow \boxed{x^0 = (0,0)}$$

$$r=0,\ s=2 \Rightarrow \boxed{x^1 = (0,2)}$$

$$r=3,\ s=0 \Rightarrow \boxed{x^2 = (3,0)}$$

$$\left.\begin{aligned} &3 - r - 2s = 0 \\ &2 - s - r = 0 \Rightarrow s = 2 - r \end{aligned}\right\} \Rightarrow \quad 3 - r - 2(2-r) = 0$$

$$3 - r - 4 + 2r = 0$$

$$-1 + r = 0$$

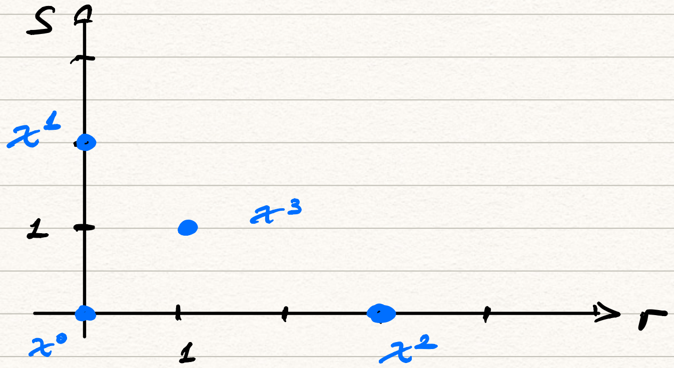
$$\Downarrow$$

$$\boxed{s = 1} \qquad \boxed{r = 1} \Rightarrow \boxed{x^3 = (1,1)}$$

Notation: Points : $x^0 = (0,0),\ x^1 = (0,2),\ x^2 = (3,0),\ x^3 = (1,1)$

Vector : $\vec{x}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ \vec{x}^1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},\ \vec{x}^2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix},\ \vec{x}^3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

Phase Space.



* Derivative Matrix and Linearizations

$$\vec{F}(x) = \begin{bmatrix} 3r\left(1-\frac{r}{3}\right) - 2rs \\ 2s\left(1-\frac{s}{2}\right) - rs \end{bmatrix} = \begin{bmatrix} 3r - r^2 - 2rs \\ 2s - s^2 - rs \end{bmatrix}$$

$$DF(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} \partial_r (3r - r^2 - 2rs) & \partial_s (3r - r^2 - 2rs) \\ \partial_r (2s - s^2 - rs) & \partial_s (2s - s^2 - rs) \end{bmatrix}$$

$$\boxed{DF(x) = \begin{bmatrix} 3 - 2r - 2s & -2r \\ -s & 2 - 2s - r \end{bmatrix}}$$ Derivative Matrix

$x^0 = (0,0)$, $x^1 = (0,2)$, $x^2 = (3,0)$, $x^3 = (1,1)$

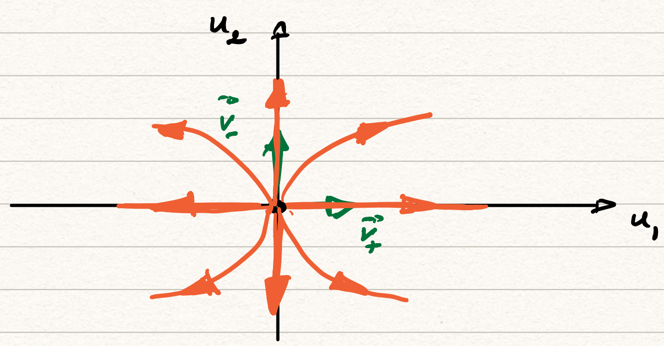
(x^0)

$DF(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$; Linearization: (at x^0) $\quad \vec{u}' = (DF(0,0))\,\vec{u}$ (Sect. 6.1)

Eigenpairs

$\lambda_+ = 3$; $\vec{V}_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_- = 2$, $\vec{V}_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$\vec{u} = \vec{0}$; Source Node

(x^1) $\quad x^1 = (0,2)$ $\qquad DF(x) = \begin{bmatrix} 3-2r-2s & -2r \\ -s & 2-2s-r \end{bmatrix}$

$$DF(0,2) = \begin{bmatrix} 3-4 & 0 \\ -2 & 2-4 \end{bmatrix}$$

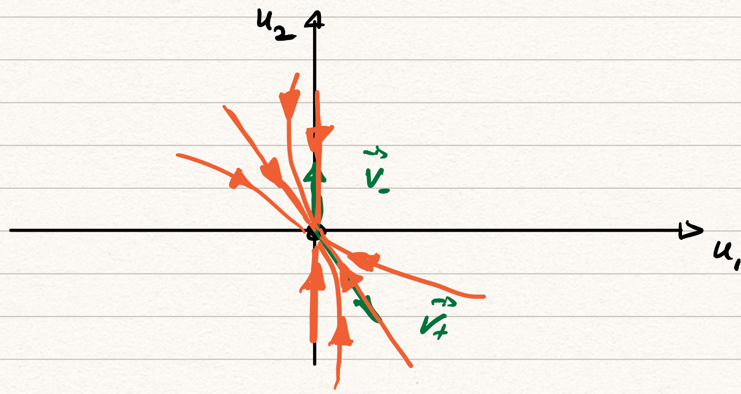
$DF(0,2) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$; $\underline{\text{Linearization:}}$ (x^1) $\qquad \vec{u}' = (DF(0,2))\,\vec{u}$

Eigenpairs $\quad$ (<u>check</u>)

$\lambda_+ = -1$, $\quad \vec{v}_+ = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda_- = -2$, $\quad \vec{v}_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{u} = \vec{0}$, $\quad$ <u>Sink Node</u>



(x^2) $\quad x^2 = (3,0)$ $\qquad DF(x) = \begin{bmatrix} 3-2r-2s & -2r \\ -s & 2-2s-r \end{bmatrix}$

$$DF(3,0) = \begin{bmatrix} 3-6-0 & -6 \\ 0 & 2-0-3 \end{bmatrix}$$

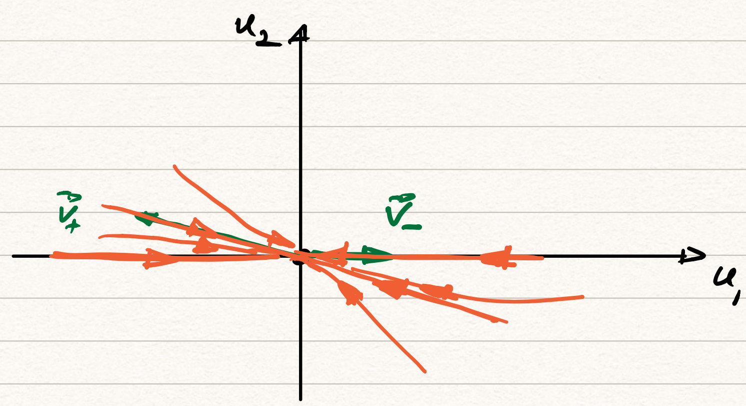
$DF(3,0) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$; $\underline{\text{Linearization}}$, (at x^3) $\qquad \vec{u}' = (DF(3,0))\,\vec{u}$

Eigenpairs : $\quad$ <u>check</u>

$\lambda_+ = -1$ $\quad \vec{v}_+ = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda_- = -3$ $\quad \vec{v}_- = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{u} = \vec{0}$ $\quad$ Sink Node



(x^3) $\quad x^3 = (1,1)$ $\qquad DF(x) = \begin{bmatrix} 3-2r-2s & -2r \\ -s & 2-2s-r \end{bmatrix}$

$DF(1,1) = \begin{bmatrix} 3-2-2 & -2 \\ -1 & 2-2-1 \end{bmatrix}$

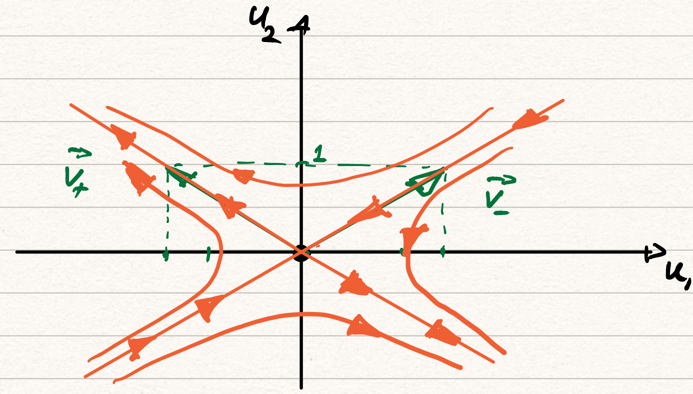
$DF(1,1) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$; Linearization (at x^3) : $\quad \vec{u}\,' = (DF(1,1))\,\vec{u}$.

Eigenpairs : <u>check</u>

$\lambda_+ = -1+\sqrt{2} > 0, \quad \vec{V}_+ = \begin{bmatrix} -\sqrt{2} \\ 1 \end{bmatrix}$

$\lambda_- = -1-\sqrt{2} < 0, \quad \vec{V}_- = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$

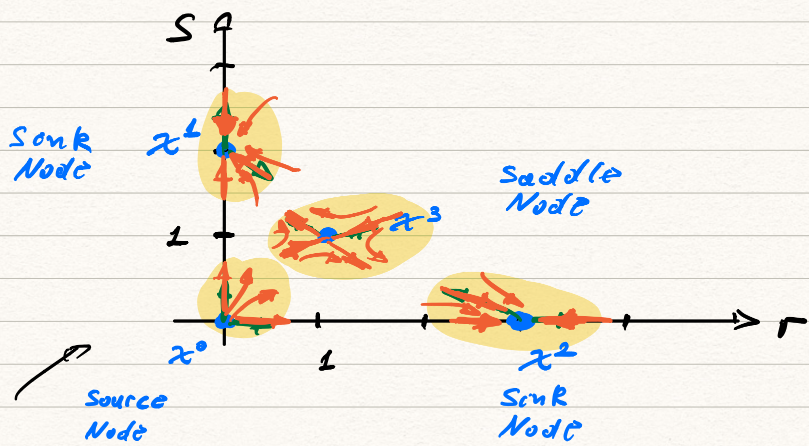
$\vec{u} = \vec{0}$ $\qquad$ Saddle Node



Hartman - Grobman : The behavior of the linearizations, $\vec{u}(t)$

same as

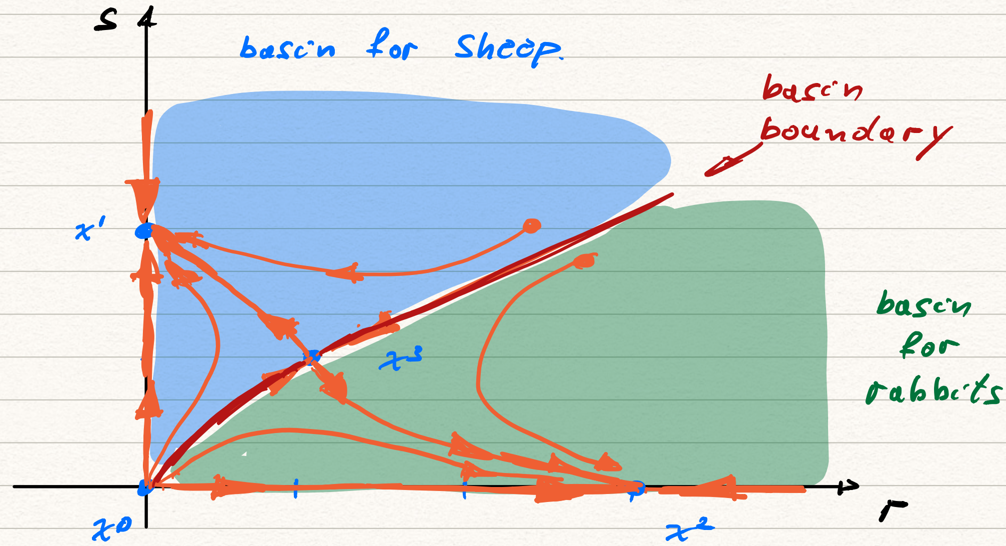
The behavior of the sols $\vec{x}(t)$ of the nonlinear system near the critical points.



Local

to

global



Model Predicts that one species goes extinct.

Example : (coexistence) Sketch the phase portrait of sols of

$$r' = r(1-r) - rs$$
$$s' = \tfrac{3}{4} s\left(1 - \tfrac{4}{3} s\right) - \tfrac{1}{2} rs$$

Sol

(1) critical Points.

$x^0 = (0,0)$, $x^1 = (0, \frac{3}{4})$, $x^2 = (1, 0)$, $x^3 = (\frac{1}{2}, \frac{1}{2})$

(2) Derivative Matrix, Linearizations

$DF(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3/4 \end{bmatrix}$ $\quad \vec{u} = \vec{0}$ Source Node

$DF(0,\frac{3}{4}) = \begin{bmatrix} 1/4 & 0 \\ -3/8 & -3/4 \end{bmatrix}$ $\quad \vec{u} = \vec{0}$ Saddle Node

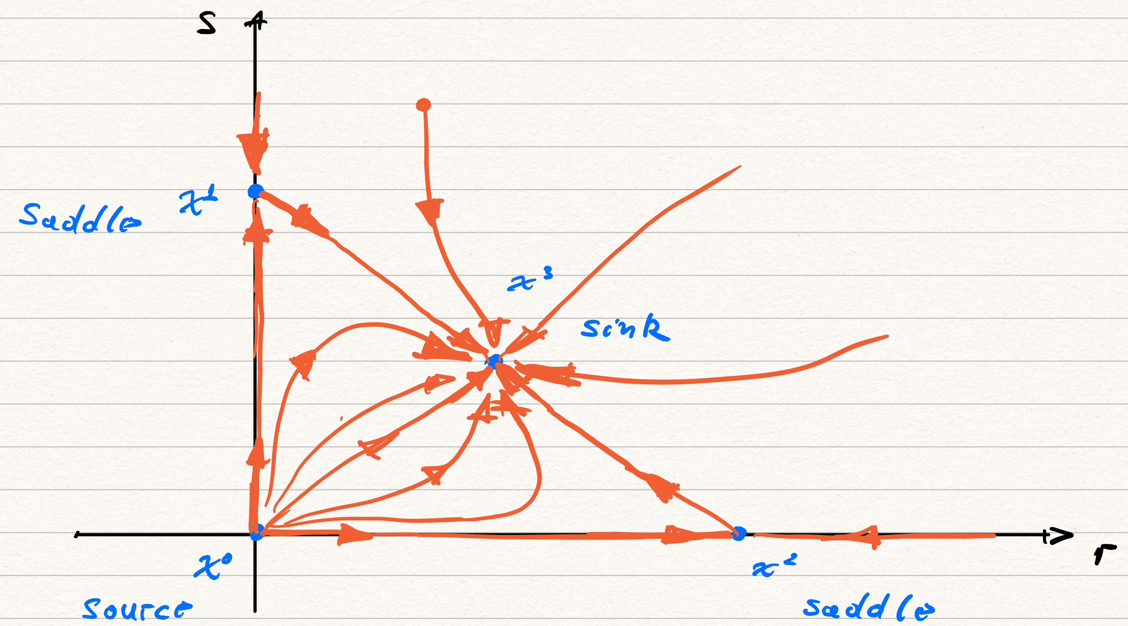
$$DF(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 1/4 \end{bmatrix} \qquad \vec{u} = \vec{0} \qquad \text{Saddle Node}$$

$$DF(\tfrac{1}{2},\tfrac{1}{2}) = \begin{bmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{bmatrix} \qquad \vec{u} = \vec{0} \qquad \text{Sink Node}$$

$$\lambda_+ = \tfrac{1}{4}\left(-2+\sqrt{2}\right) < 0$$

$$\lambda_- = \tfrac{1}{4}\left(-2-\sqrt{2}\right) < 0$$

Phase Space



Model predicts coexistence of rabbits and sheep.