Section 6.3: 2x2 Nonlinear Systems

Plan:
- Review: 2x2 Linear Systems ✓
- Critical Points and Linearizations ←
- Example

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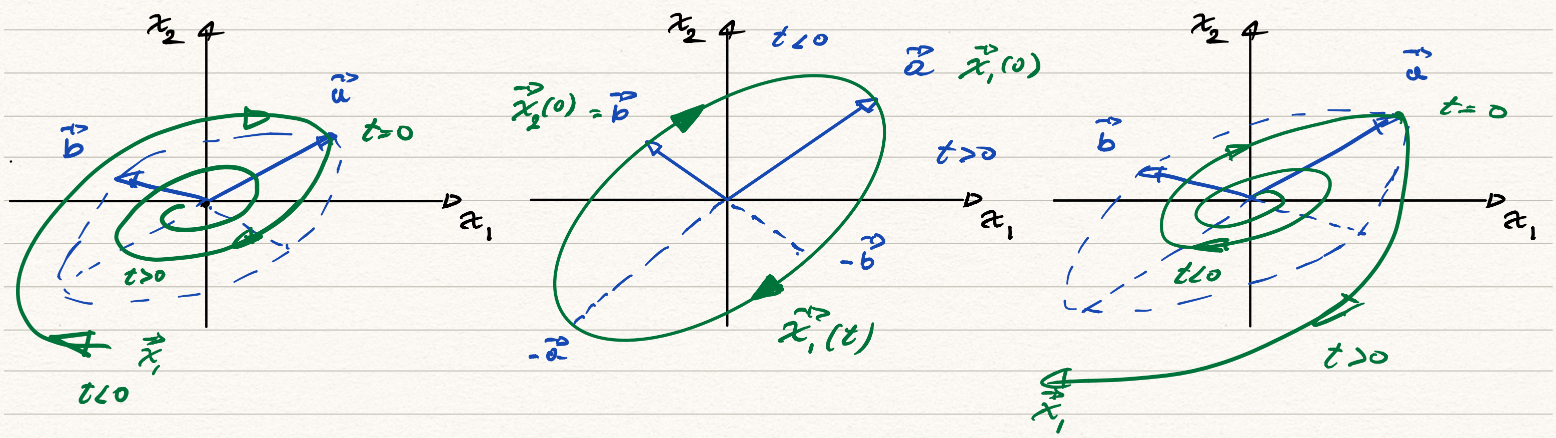
Review: 6.2 : complex case

$\boxed{\lambda_{\pm} = \alpha \pm i\beta}$ $\boxed{\vec{v}_{\pm} = \vec{a} \pm \vec{b}\, i}$



$\boxed{\alpha < 0}$	$\boxed{\alpha = 0}$	$\boxed{\alpha > 0}$
$\vec{x} = \vec{0}$ Sink Spiral	$\vec{x} = 0$ center	$\vec{x} = 0$ Source spiral
Stable	Neither stable nor unstable.	unstable

$\vec{x}_1(t)$ t increases

$\vec{a} \rightarrow -\vec{b}$

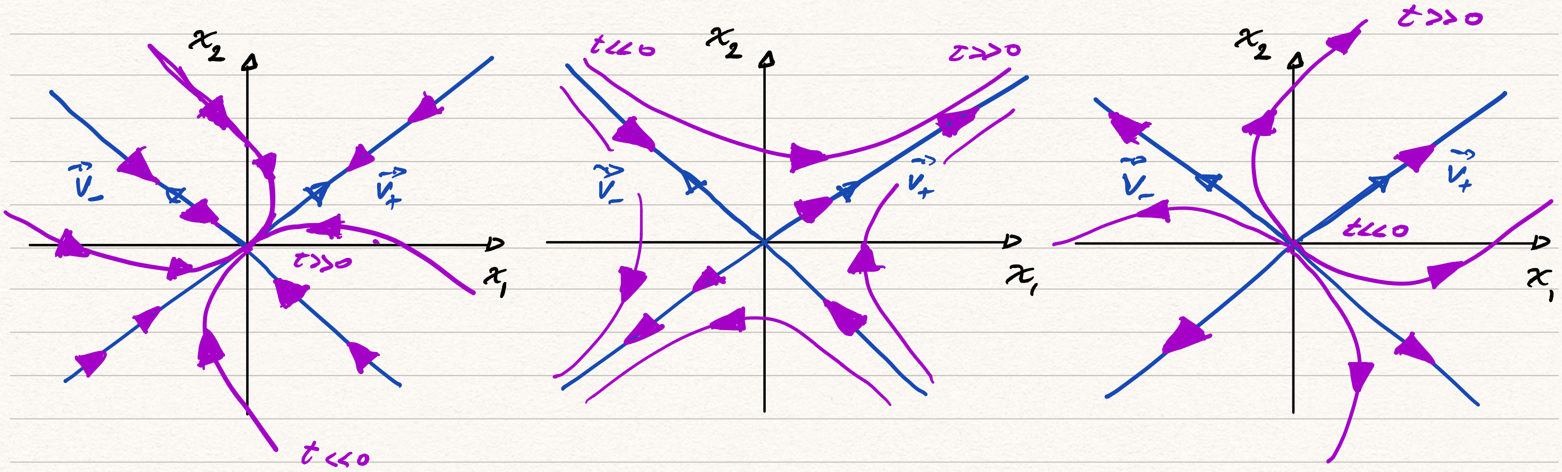
$\vec{x}_2$ t increases

$\vec{b} \rightarrow \vec{a}$

($\vec{a} \rightarrow -\vec{b}$ same spin direction as $\vec{b} \rightarrow \vec{a}$)

6.2 : Real Different case.

$\boxed{\lambda_+ \neq \lambda_- \ , \ \text{Reals}}$ $\quad \boxed{\vec{V}_+ , \vec{V}_-}$



$\lambda_- < \lambda_+ < 0$	$\lambda_- < 0 < \lambda_+$	$0 < \lambda_- < \lambda_+$
$\vec{x} = 0$ Sink Node	$\vec{x} = 0$ Saddle Node	$\vec{x} = 0$ Source Node
Stable	Unstable	Unstable

* Critical Points and Linearizations

Problem (autonomous): [Sketch a Phase Portrait of solutions of

$$\vec{x}'(t) = \vec{F}(x(t)) \qquad (1)$$

]

Remark: We do not solve (1). It's too difficult.

Plan: (1) Find equilibrium sols, x^e, of (1)

↳ also called: critical points

(2) We study the behavior of sols $x(t)$ of (1) when they are near x^e.

(3) We introduce:

$\vec{x}^e$: Vector from origin ending at x^e

$\vec{x}(t)$: " " " " " $x(t)$

$\vec{u}(t) = \vec{x}(t) - \vec{x}^e$ ← Change in $\vec{x}$ from the equil. sol. $\vec{x}^e$.

(4) We find an equation for this change, $\vec{u}(t)$, when this change is small.

(5) This equation for $\vec{u}$ will be called the linearization of (1) at $\vec{x}^e$.

(6) Later on we will see when the solutions of this linearization are a good approximation of solutions of (1).

* Finding the Linearization of (1)

We start $\vec{x}' = \vec{F}(\vec{x})$ (1)

We have $\vec{x}^c$: Equil. Sol. (critical point)

$\vec{F}(\vec{x}^c) = \vec{0}$ $(\vec{x}^{c\prime} = \vec{0})$

we introduce $\boxed{\vec{u}(t) = \vec{x}(t) - \vec{x}^c}$ $\Rightarrow \boxed{\vec{x}(t) = \vec{u}(t) + \vec{x}^c}$

we find an eq. for $\vec{u}$.

$$\vec{u}' = \vec{x}' - \underbrace{\vec{x}^{c\prime}}_{=0} = \vec{F}(\vec{x}) = \vec{F}(\vec{u} + \vec{x}^c)$$

$$\boxed{\vec{u}' = \vec{F}(\vec{u} + \vec{x}^c)} \qquad (1)$$

In components: $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$; $\vec{F} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$, $\vec{x}^c = \begin{bmatrix} x_1^c \\ x_2^c \end{bmatrix}$

$$\boxed{\begin{aligned} u_1' &= f_1((u_1 + x_1^c), (u_2 + x_2^c)) \\ u_2' &= f_2((u_1 + x_1^c), (u_2 + x_2^c)) \end{aligned}} \qquad (1)$$

when $\vec{u}$ is small , $|u_1| \ll 1$, $|u_2| \ll 1$,

We can compute the Taylor expansions of f_1, f_2 around x_1^c, x_2^c

$$u_1' = \underbrace{\cancel{f_1(x_1^c, x_2^c)}}_{=0} + \left.\frac{\partial f_1}{\partial x_1}\right)_{x^c} u_1 + \left.\frac{\partial f_1}{\partial x_2}\right)_{x^c} u_2 + \mathcal{O}(u_1^2, u_2^2, u_1 u_2)$$

$$u_2' = \underbrace{\cancel{f_2(x_1^c, x_2^c)}}_{=0} + \left.\frac{\partial f_2}{\partial x_1}\right)_{x^c} u_1 + \left.\frac{\partial f_2}{\partial x_2}\right)_{x^c} u_2 + \mathcal{O}(u_1^2, u_2^2, u_1 u_2) \qquad (1)$$

$\hookrightarrow \vec{F}(\vec{x}^c) = \vec{0}$

<u>Notation</u> :

$$\left.\begin{aligned} \left.\frac{\partial f_1}{\partial x_1}\right)_{x^c} &= \partial_1 f_1 & \left.\frac{\partial f_1}{\partial x_2}\right)_{x^c} &= \partial_2 f_1 \\ \left.\frac{\partial f_2}{\partial x_1}\right)_{x^c} &= \partial_1 f_2 & \left.\frac{\partial f_2}{\partial x_2}\right)_{x^c} &= \partial_2 f_2 \end{aligned}\right\} \underline{\text{Numbers}}$$

$$u_1' = (\partial_1 f_1)\, u_1 + (\partial_2 f_1)\, u_2 + \underbrace{O(u_1^2, u_2^2, u_1 u_2)}$$
$$u_2' = (\partial_1 f_2)\, u_1 + (\partial_2 f_2)\, u_2 + \underbrace{O(u_1^2, u_2^2, u_1 u_2)} \qquad (1)$$

If $\vec{u}$ is small, then are small.

$$\boxed{\begin{aligned} u_1' &= (\partial_1 f_1)\, u_1 + (\partial_2 f_1)\, u_2 \\ u_2' &= (\partial_1 f_2)\, u_1 + (\partial_2 f_2)\, u_2 \end{aligned}} \qquad (2)$$

Eq. (2) is different from Eq. (1)

Eq. (2) is the linearization of Eq. (1) at $\vec{x}^c$.

Eq. (2) is a 2x2 linear diff. eq.

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad (DF_c) = \begin{bmatrix} (\partial_1 f_1) & (\partial_2 f_1) \\ (\partial_1 f_2) & (\partial_2 f_2) \end{bmatrix}$$ const. Matrix.

$\boxed{\vec{u}' = (DF_c)\, \vec{u}}$ is the linearization of $\vec{x}' = \vec{F}(\vec{x})$ at $\vec{x}^c$.

Eq (2)

$\downarrow$

We know how to solve for $\vec{u}$. (Sect. 6.1)

Hartman-Grobman: The sols $\vec{u}$ of Eq. (2) are a good approximation of the sols $\vec{x}$ of Eq. (1) near $\vec{x}^c$ when matrix (DF_c) has eigenvalues with nonzero real parts.

Def : The linearization of $\vec{x}' = \vec{F}(x)$ at an equilibrium sol $\vec{x}^c$ is a linear system

$$\vec{u}' = (DF_c)\,\vec{u} \,,$$

where the derivative matrix of $\vec{F}$ at $\vec{x}^c$ is

$$(DF_c) = \begin{array}{cc} \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right)_{\vec{x}^c} & \left.\dfrac{\partial f_1}{\partial x_2}\right)_{\vec{x}^c} \\[2em] \left.\dfrac{\partial f_2}{\partial x_1}\right)_{\vec{x}^c} & \left.\dfrac{\partial f_2}{\partial x_2}\right)_{\vec{x}^c} \end{bmatrix} & \begin{matrix} \leftarrow f_1 \\[3em] \leftarrow f_2 \end{matrix} \\ \begin{matrix} \uparrow & \qquad\qquad & \uparrow \\ \partial x_1 & \qquad\qquad & \partial x_2 \end{matrix} & \end{array}$$

<u>**Notation**</u> :

$DF(x)$: Derivative matrix at any x.

$DF(x)$: Also called Jacobian Matrix.

$DF_c = DF(x^c)$: Derivative matrix at x^c

* *Example : Sketch a Phase Portrait*

Example : Find all critical points of

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2x_2$$

decoupled.

Sol

$$\vec{F} = \begin{bmatrix} -x_1 + (x_1)^3 \\ -2x_2 \end{bmatrix} \quad ; \quad \vec{F}(\vec{x}^c) = \vec{0} \quad \Rightarrow$$

$$-x_1 + (x_1)^3 = 0 \quad \Rightarrow \quad x_1\left((x_1)^2 - 1\right) = 0 \quad \Rightarrow \quad x_1 = 0 \quad (a)$$

or

$$(x_1)^2 - 1 = 0 \quad (b)$$

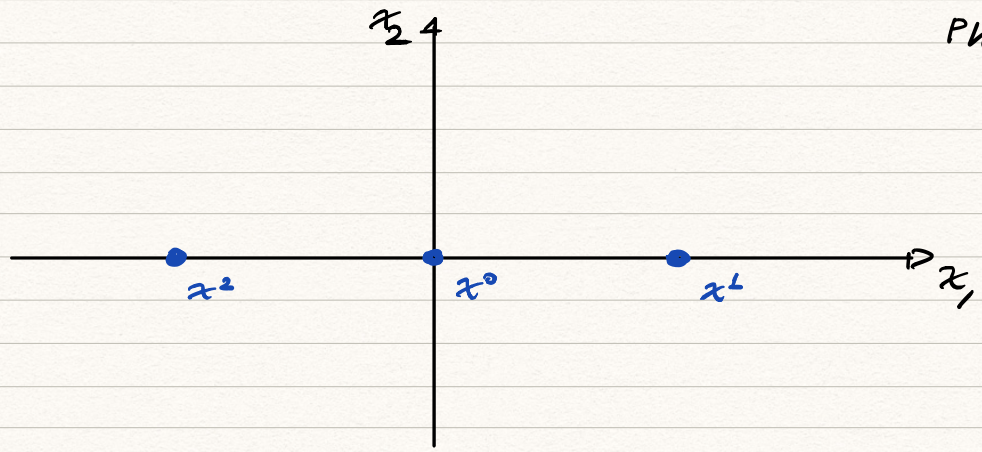
$$-2x_2 = 0 \quad \Rightarrow \quad \boxed{x_2 = 0}$$

(a) $\boxed{x^0 = (0,0)}$, (b) $x_1^2 = 1 \Rightarrow x_1 = \pm 1$

$$\boxed{\begin{aligned} x^1 &= (1,0) \\ x^2 &= (-1,0) \end{aligned}}$$

$$\boxed{\vec{x}^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \vec{x}^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{x}^2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}}$$

Phase Space



Example: Find the linearization at every $\vec{x}^e$ of

$$x_1 = -x_1 + (x_1)^3$$
$$x_2 = -2x_2$$

Sol

$$\vec{F}(\vec{x}) = \begin{bmatrix} -x_1 + (x_1)^3 \\ -2x_2 \end{bmatrix}$$

The Derivative Matrix

$$F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \quad ; \quad DF(x) = \begin{array}{c} \begin{array}{cc} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \\ \downarrow & \downarrow \end{array} \\ \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \end{array} \begin{array}{l} \leftarrow f_1 \\ \\ \leftarrow f_2 \end{array}$$

$$DF(x) = \begin{bmatrix} \frac{\partial}{\partial x_1}(-x_1 + x_1^3) & \frac{\partial}{\partial x_2}(-x_1 + x_1^3) \\ \frac{\partial}{\partial x_1}(-2x_2) & \frac{\partial}{\partial x^2}(-2x_2) \end{bmatrix}$$

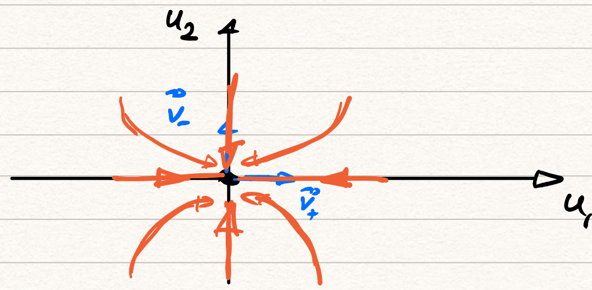
$$DF(x) = \begin{bmatrix} (-1 + 3x_1^2) & 0 \\ 0 & -2 \end{bmatrix}$$ Derivative Matrix (Jacobian)

$x^0 = (0,0)$, $x^1 = (1,0)$, $x^2 = (-1,0)$

at $x^0 = (0,0)$ $DF_0 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$; $\vec{u}' = DF_0 \, \vec{u}$

eigenpairs $\lambda_+ = -1$ $\vec{v}_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\lambda_- = -2$ $\vec{v}_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

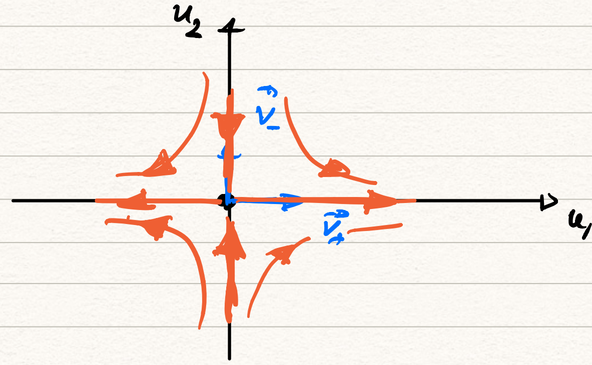


$\vec{u} = \vec{0}$ Sink Node.

$$\left.\begin{array}{l} \text{at } \underline{x^1 = (1,0)} \\ \quad\; \underline{x^2 = (-1,0)} \end{array}\right\} \quad DF_1 = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \vec{u} = DF_1 \, \vec{u}$$

eigenpairs $\quad \lambda_+ = 2, \quad \vec{v}_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\qquad\qquad\quad\; \lambda_- = -2 \quad \vec{v}_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



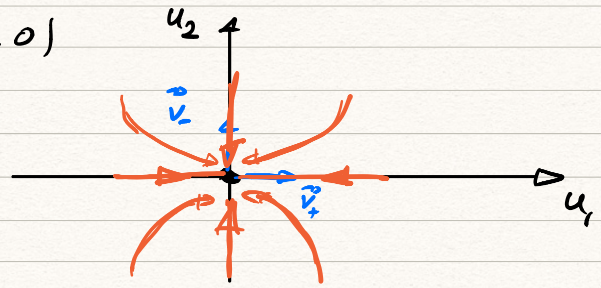
$\vec{u} = \vec{0}$ Saddle Node

Example : Sketch a phase portrait of

$$x_1' = -x_1 + (x_1)^3$$
$$x_2' = -2x_2$$

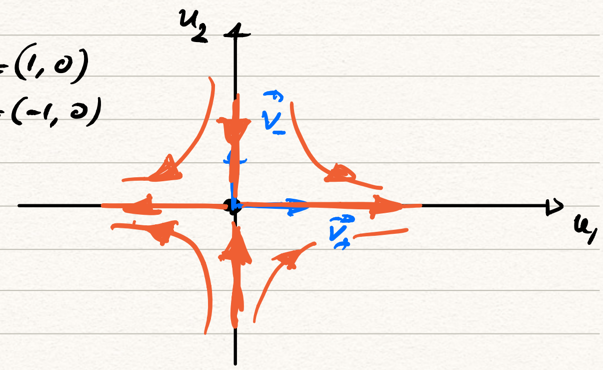
Sol

$x^0 = (0,0)$



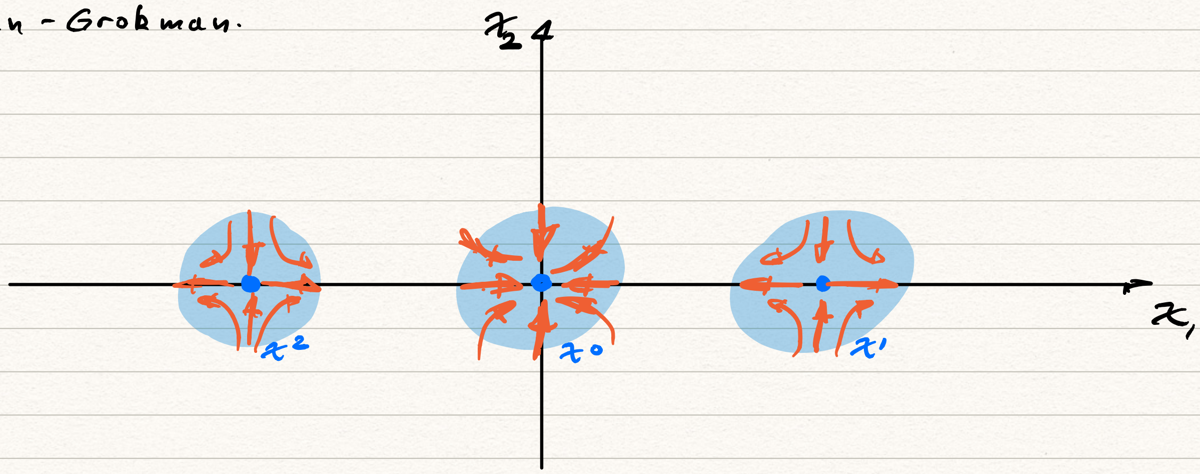
$\vec{u} = \vec{0}$ Sink Node.

$x^1 = (1,0)$
$x^2 = (-1,0)$

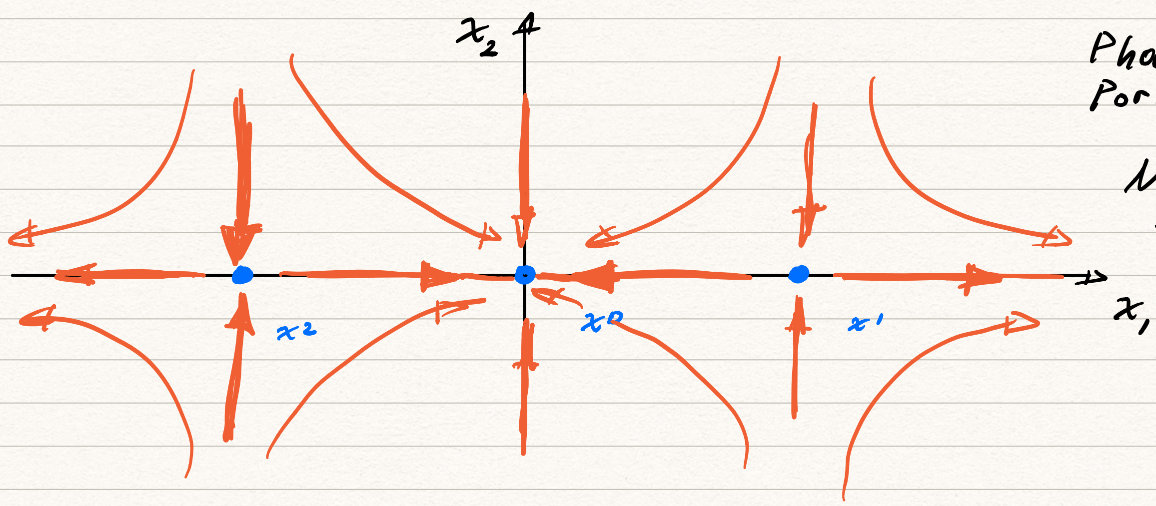


$\vec{u} = \vec{0}$ Saddle Node

Hartman-Grobman.



Saddle Node Sink Node Saddle Node



Phase Portrait.

Non-Linear Syst.