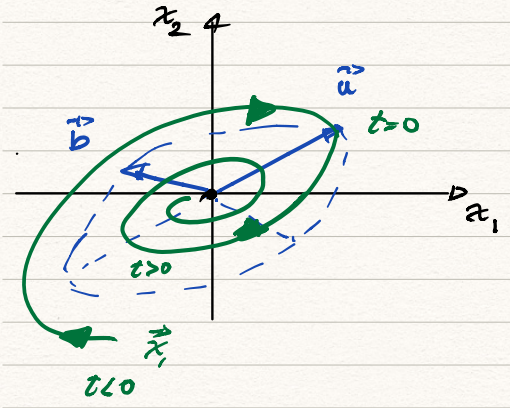
Section 6.2: 2×2 Phase Portraits and Classification

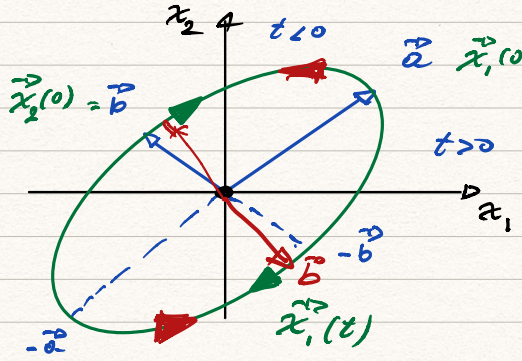
Plan:
* Overview of Phase Portraits $\vec{x}' = A\vec{x}$ ✓
* Complex Eigenvalues ✓
* Real Different Eigenvalues ←

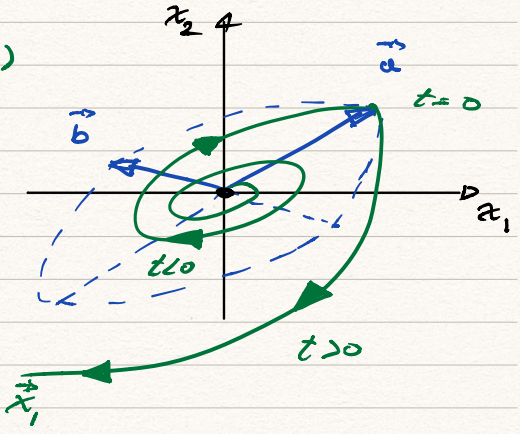
—o—

* Review: A Has Complex Eigenpairs

$$\lambda_1 = \alpha \pm i\beta \qquad \vec{v}_+ = \vec{a} \pm \vec{b}\, i$$







$\alpha < 0$	$\alpha = 0$	$\alpha > 0$
$\vec{x} = \vec{0}$ Sink Spiral	$\vec{x} = 0$ center	$\vec{x} = 0$ Source spiral
Stable	Neither stable nor unstable.	Unstable

$\vec{x}_1(t)$ t increases

$\vec{a} \to -\vec{b}$

$\vec{x}_2$ t increases

$\vec{b} \to \vec{a}$

($\vec{a} \to -\vec{b}$ Same spin direction as $\vec{b} \to \vec{a}$)

* Real Different Eigenvalues

Overview: $\vec{x}' = A\vec{x}$; A has eigenvalues $\lambda_+ \neq \lambda_-$, reals.

Cases:

(1) $\lambda_- < \lambda_+ < 0$ ✓

(2) $\lambda_- < 0 < \lambda_+$ ←

(3) $0 < \lambda_- < \lambda_+$ ✓

Example: Sketch a P.P. of sols of $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} -2 & -3 \\ -3 & -2 \end{bmatrix}$ (L)

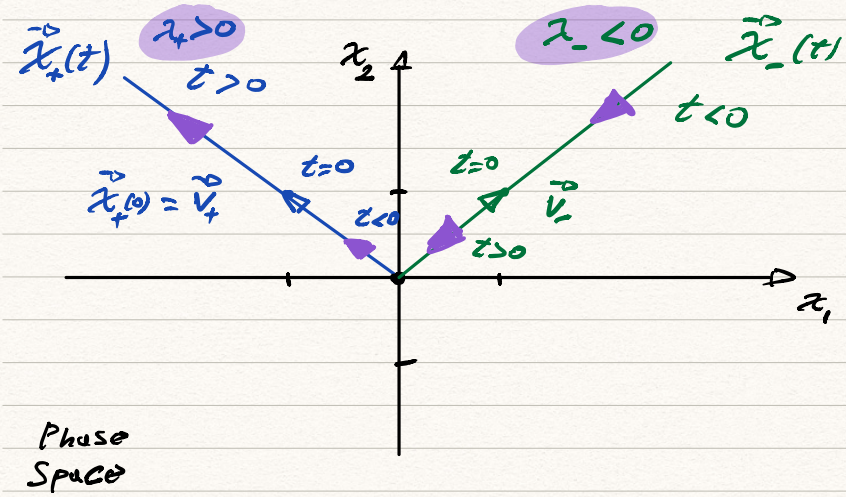
Sol:

Eigenpairs of A:

$$\left\{ \lambda_+ = 1, \ \vec{v}_+ = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad , \quad \lambda_- = -5, \ \vec{v}_- = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \Rightarrow \lambda_- < 0 < \lambda_+$$

Fund. Sols of (L)

$$\vec{x}_+(t) = e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \vec{x}_-(t) = e^{-5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Phase Space

$\vec{x}_+(t) = e^{t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{x}_+(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v}_+$

$t > 0$, $e^{t} > 1$

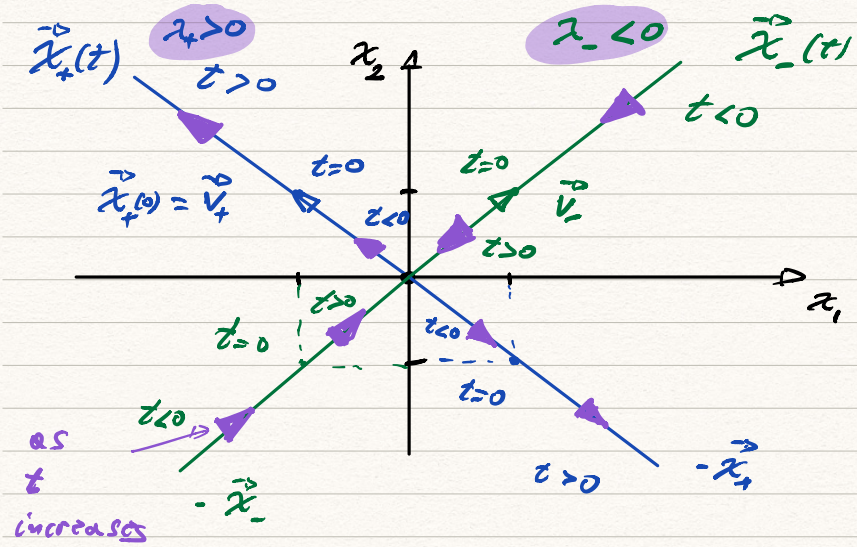
$t < 0$, $0 < e^{t} < 1$

$\vec{x}_-(t) = e^{-5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{x}_-(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{v}_-$

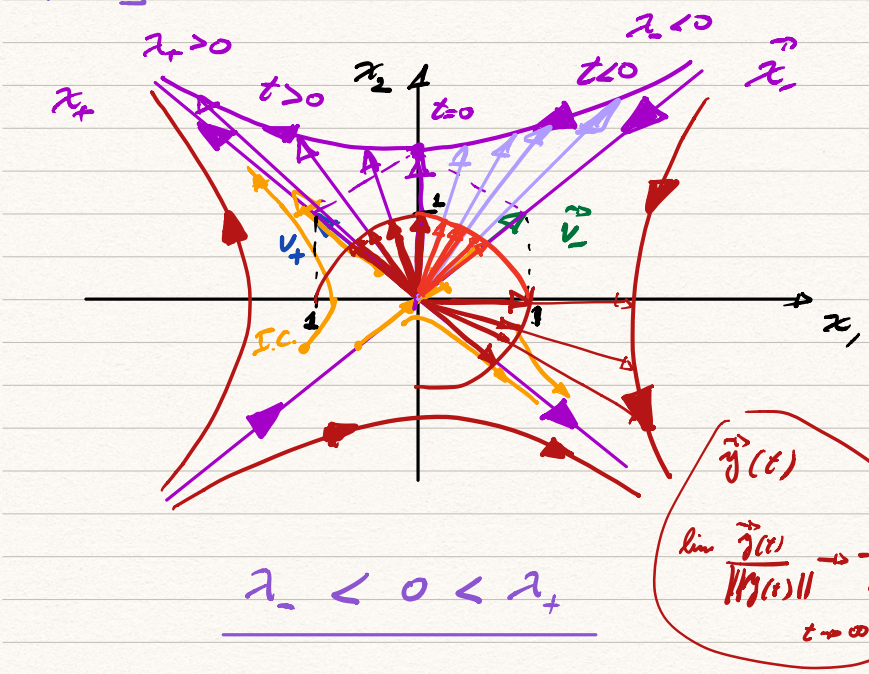
$t > 0$, $0 < e^{-5t} < 1$

$t < 0$, $e^{-5t} > 1$



$$\vec{x}_+(t) = e^{t}\,\vec{v}_+$$
$$\vec{x}_-(t) = e^{-5t}\,\vec{v}_-$$

$\vec{x}_+(t)$	$\vec{x}_-(t)$
$-\vec{x}_+(t)$	$-\vec{x}_-(t)$



$$\lambda_- < 0 < \lambda_+$$

$\vec{y}(t)$

$$\lim_{t \to \infty} \frac{\vec{y}(t)}{\|\vec{y}(t)\|} \to -\frac{\vec{v}_+}{\|\vec{v}_+\|}$$

$$\vec{x}(t) = \vec{x}_+(t) + \vec{x}_-(t)$$

$$\vec{x}(0) = \vec{v}_+ + \vec{v}_-$$

$$\vec{x}(t) = e^{t}\,\vec{v}_+ + e^{-5t}\,\vec{v}_-$$

$t > 0$, $e^{t} > 1$, $0 < e^{-5t} < 1$

$t < 0$, $0 < e^{t} < 1$, $e^{-5t} > 1$

Remarks (Homework)

$$\vec{x}(t) = \vec{x}_+(t) + \vec{x}_-(t)$$

(1) Magnitude :

$$\begin{cases} \|\vec{x}(t)\| \xrightarrow[t \to \infty]{} \infty \\ \|\vec{x}(t)\| \xrightarrow[t \to -\infty]{} \infty \end{cases}$$

(2) Direction :

$$\tilde{\vec{x}}(t) = \frac{\vec{x}(t)}{\|\vec{x}(t)\|}$$ unit vector

$$\tilde{\vec{x}}(t) \xrightarrow[t \to \infty]{} \frac{\vec{v}_+}{\|\vec{v}_+\|}$$

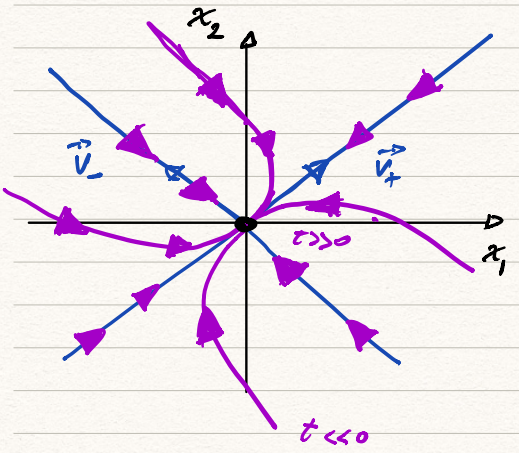
$\tilde{\vec{x}}(t) \to \frac{\vec{v}_-}{\|\vec{v}_-\|}$

$t \to -\infty$

Overview:

$\lambda_+ \neq \lambda_-$, Reals

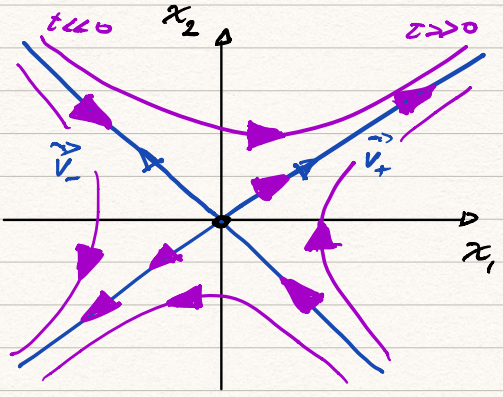
$\vec{v}_+$, $\vec{v}_-$



$\lambda_- < \lambda_+ < 0$

$\vec{x}=0$ Sink Node

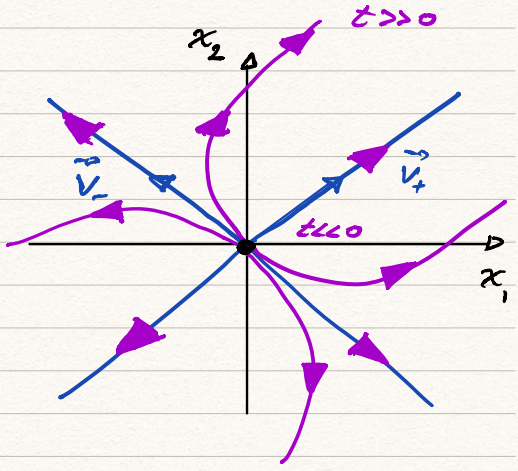
Stable



$\lambda_- < 0 < \lambda_+$

$\vec{x}=0$ Saddle Node

Unstable



$0 < \lambda_- < \lambda_+$

$\vec{x}=0$ Source Node

Unstable