Section 6.2: 2x2 Phase Portraits and Classification

Plan:
* Overview of Phase Portraits ←
* Complex Eigenvalues
* Real Different Eigenvalues

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* Overview of Phase Portraits

Review of Solution Formulas for 2x2 SFOLDE

Thrm: Fundamental sols. of $\vec{x}' = A\vec{x}$, with A 2x2, real, with eigenpairs $\lambda_+, \vec{v}_+$, and $\lambda_-, \vec{v}_-$, are:

(a) $\lambda_+ \neq \lambda_-$, real, (A diagonalizable)

$$\vec{x}_+ = e^{\lambda_+ t}\,\vec{v}_+ , \qquad \vec{x}_- = e^{\lambda_- t}\,\vec{v}_- .$$

(b) $\lambda_\pm = \alpha \pm i\beta$, $\vec{v}_\pm = \vec{a} \pm i\vec{b}$, ($A$ diagonalizable)

$$\vec{x}_1 = e^{\alpha t}\left(\cos(\beta t)\,\vec{a} - \sin(\beta t)\,\vec{b}\right)$$

$$\vec{x}_2 = e^{\alpha t}\left(\sin(\beta t)\,\vec{a} + \cos(\beta t)\,\vec{b}\right)$$

(c) $\lambda_+ = \lambda_- = \lambda_0$, $A = \lambda_0 I$,

$$\vec{x}_+ = e^{\lambda_0 t}\begin{bmatrix}1\\0\end{bmatrix}, \qquad \vec{x}_- = e^{\lambda_0 t}\begin{bmatrix}0\\1\end{bmatrix}.$$

(d) $\lambda_+ = \lambda_- = \lambda_0$, A not diagonalizable,

$$\vec{x}_1 = e^{\lambda_0 t}\,\vec{v}, \qquad \vec{x}_2 = e^{\lambda_0 t}\left(t\vec{v} + \vec{w}\right),$$

where: $(A - \lambda_0 I)\vec{v} = \vec{0}$

$(A - \lambda_0 I)\vec{w} = \vec{v}$.

<u>Example</u>: Sketch a phase portrait, (diagram) of sols of

$$\underline{\vec{x}' = A\vec{x}}, \qquad A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

<u>Sol</u>

Eigenvals. of A. $\quad 0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 2 \\ -2 & -\lambda \end{vmatrix} = \underline{\lambda^2 + 4}$

$$\boxed{\lambda_\pm = \underline{\pm 2i}} \qquad \lambda_\pm = \alpha \pm i\beta$$

$$\boxed{\alpha = 0, \quad \beta = 2}$$

Eigenvect. $\xrightarrow[\text{exercise}]{}$ $\boxed{\vec{v}_+ = \begin{bmatrix} 1 \\ i \end{bmatrix}}$; $\boxed{\vec{v}_- = \overline{\vec{v}_+} = \begin{bmatrix} 1 \\ -i \end{bmatrix}}$

$$\vec{v}_+ = \vec{a} + i\vec{b}$$

$$\vec{v}_+ = \begin{bmatrix} 1 + 0i \\ 0 + 1i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{a}} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}} i$$

$$\vec{x}_1 = e^{\alpha t}\left(\cos(\beta t)\,\vec{a} - \sin(\beta t)\,\vec{b}\right)$$

$$\vec{x}_1(t) = \underset{1}{e^{0t}}\left(\cos(2t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

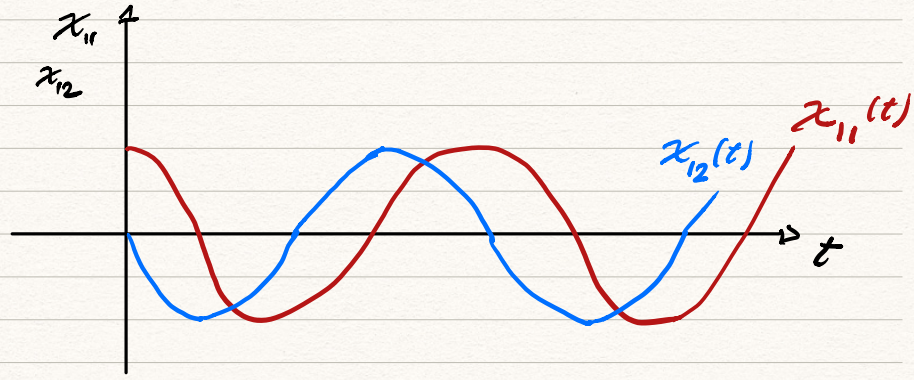
$$\boxed{\vec{x}_1(t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}} \xrightarrow{\text{Analogously}} \boxed{\vec{x}_2(t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}$$

(1) Graph Components

$$\vec{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{12}(t) \end{bmatrix}$$

$$x_{11}(t) = \cos(2t)$$

$$x_{12}(t) = -\sin(2t)$$



(2) Phase Portrait (diagram)

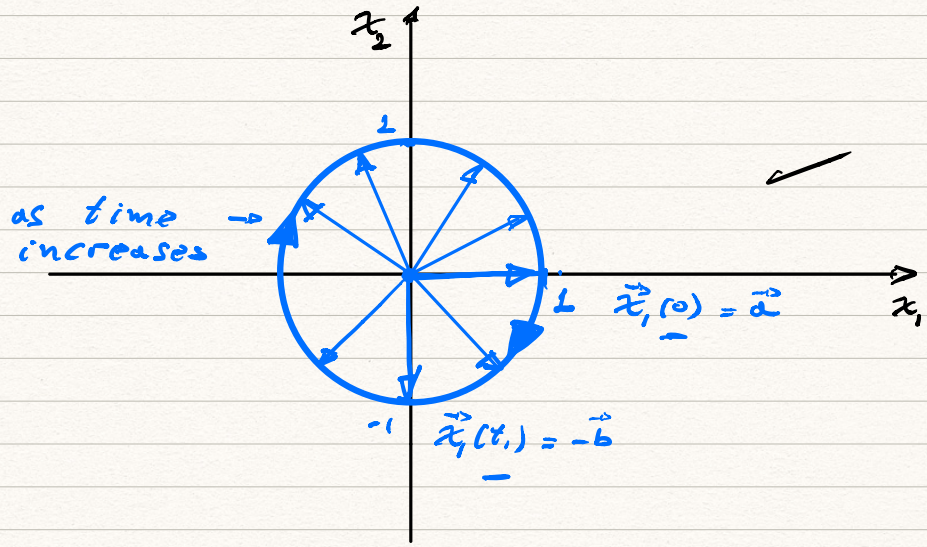
(Phase Space)

$$\underline{\vec{x}_1(t)} = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

$$\vec{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a}$$

$$\vec{x}_1\left(2t_1 = \frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\vec{b}$$

$$t_1 = \frac{\pi}{4}$$



Length of $\vec{x}_1(t)$: $\|\vec{x}_1(t)\| = \sqrt{(x_{11}(t))^2 + (x_{12}(t))^2}$

$$= \sqrt{\cos^2(2t) + (-1)^2 \sin^2(2t)}$$

$$= \underline{1}$$

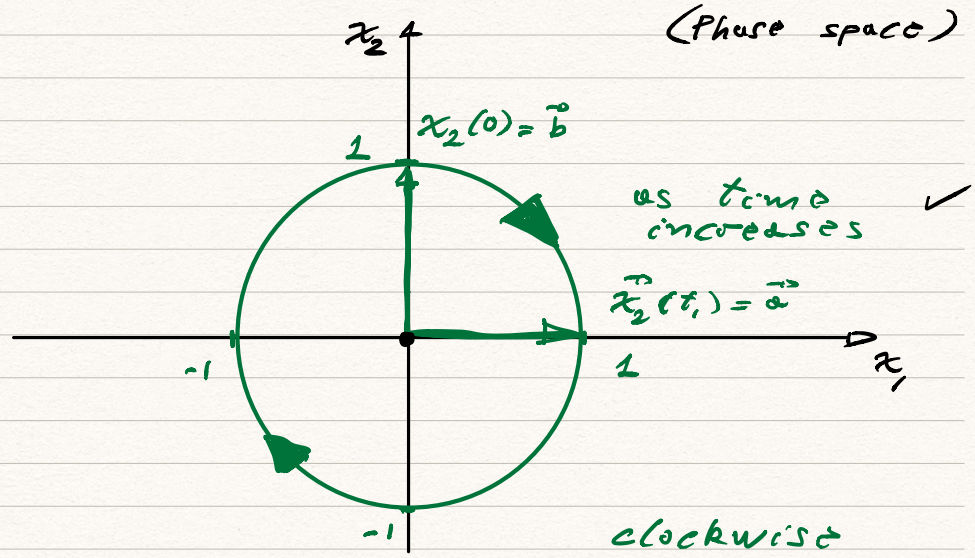
$$\boxed{\vec{x}_2(t) = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}} \Rightarrow \boxed{\|\vec{x}_2(t)\| = 1}$$

$$\vec{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \vec{b}$$

$$\vec{x}_2\left(2t_1 = \frac{\pi}{2}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a}$$

$$t_1 = \frac{\pi}{4}$$

(Phase space)



Remark

$\boxed{\alpha = 0}$ $\boxed{\text{Center}}$

All solutions are closed curves around the trivial solution $\vec{x}(t) = \vec{0}$ all t.

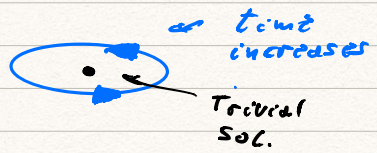
$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

* Complex Eigenvalues

Remark: $\vec{x}' = Ax$, A has complex eigenvalues

$\lambda_\pm = \alpha \pm i\beta$, $\beta > 0$.

- $\alpha = 0$



Center

- $\alpha < 0$

Sink Spiral.

* $\alpha > 0$

Source Spiral.

Example: [Sketch a phase portrait of Fund. Sols of $\vec{x}' = A\vec{x}$, $A = \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$]

Sol

Eigenpairs of A:

$$\boxed{\begin{aligned} \lambda_\pm &= 3 \pm 2i \\ \vec{v}_\pm &= \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \end{aligned}}$$

Exercise: Show

$$\vec{v}_+ = \begin{bmatrix} 1+0i \\ 0+1i \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\vec{a}} + i \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\vec{b}}$$

$$\vec{x}_1 = e^{\alpha t}\left(\cos(\beta t)\, \vec{a} - \sin(\beta t)\, \vec{b} \right)$$

$$\vec{x}_2 = e^{\alpha t}\left(\sin(\beta t)\, \vec{a} + \cos(\beta t)\, \vec{b} \right)$$

$$\vec{x}_1(t) = e^{3t}\left(\cos(2t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(2t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

$$\boxed{\vec{x}_1(t) = e^{3t} \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}}$$

Analogously: $\boxed{\vec{x}_2(t) = e^{3t} \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}}$

Phase Portrait

$$\vec{x}_1(t) = e^{3t}\, \tilde{\vec{x}}_1(t)$$

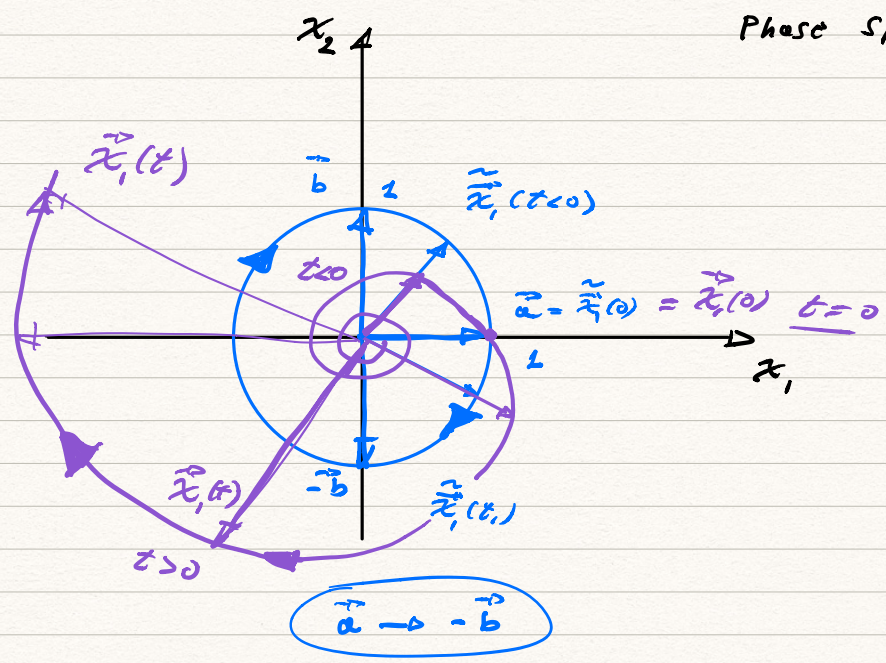
$$\tilde{\vec{x}}_1(t) = \begin{bmatrix} \cos(2t) \\ -\sin(2t) \end{bmatrix}$$

$$\tilde{\vec{x}}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{a}$$

$$\tilde{\vec{x}}_1\left(2t_1 = \tfrac{\pi}{2}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\vec{b}$$

$$\vec{x}_1(0) = 1\, \tilde{\vec{x}}_1(0)$$

Phase Space



$\underline{t>0}$; $e^{3t} > 1$; $\vec{x}_1(t)$ longer than $\tilde{\vec{x}}_1(t)$

$\underline{t<0}$; $0 < e^{3t} < 1$; $\vec{x}_1(t)$ shorter than $\tilde{\vec{x}}_1(t)$

<u>Remarks</u> (Webwork)

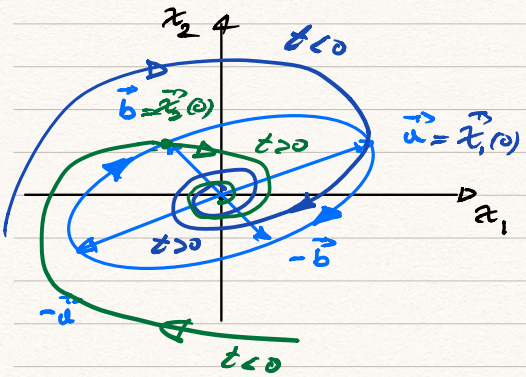
(1) $\|\vec{x}_1(t)\| \xrightarrow[t\to\infty]{} \underline{\infty}$ Because $\underline{\alpha > 0}$ $(\alpha = 3)$

$\|\vec{x}_1(t)\| \xrightarrow[t\to-\infty]{} \underline{0}$ Because $\underline{\alpha>0} \Rightarrow \underline{e^{\alpha t} \xrightarrow[t\to-\infty]{} 0}$

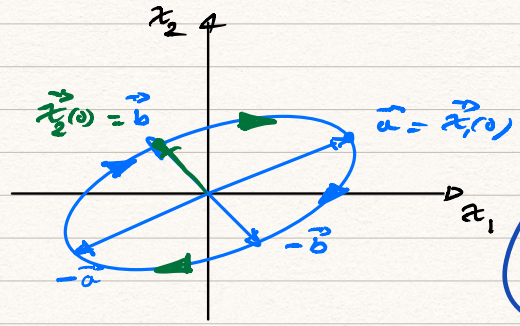
(2) $\vec{u}_1(t) = \dfrac{\vec{x}_1(t)}{\|\vec{x}_1(t)\|}$ <u>unit</u> <u>Vector</u> in the direction of $\vec{x}_1(t)$

$\dfrac{\vec{x}_1(t)}{\|\vec{x}_1(t)\|} \xrightarrow[t\to\infty,\ t\to-\infty]{} \underline{DNE}$ Spins Forever.

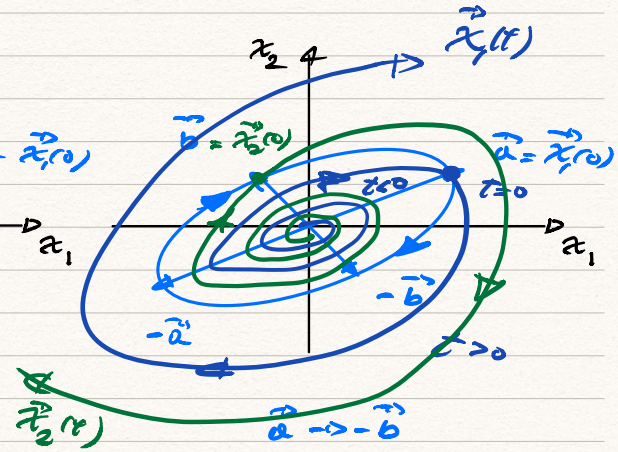
Overview: $\boxed{\lambda_\pm = \alpha \pm i\beta}$ $\boxed{\vec{v}_\pm = \vec{a} \pm \vec{b}\, i}$



$\vec{a} \to -\vec{b}$



$\vec{a} \to -\vec{b}$



$\alpha < 0$	$\alpha = 0$	$\alpha > 0$
$\vec{x} = \vec{0}$ Sink spiral	$\vec{x} = 0$, center	$\vec{x} = 0$ Source spiral
stable		unstable

$\vec{x}_1(t)$ t increases

$\vec{a} \to -\vec{b}$

$\vec{x}_2$ t increases

$\vec{b} \to \vec{a}$