Section 5.5: Diagonalizable Matrices

Plan: * Example: (Non) Diagonalizable Matrices

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Example: Show that A is diagonalizable but B is not.

$$A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Sol:

Eigenvalues: (For both A and B)

$$\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda)^2(1-\lambda) = \det(B-\lambda I)$$

upper triangular

$\boxed{\lambda_1 = 1}$, $\boxed{\lambda_2 = 3}$ eigenvalues to both A and B.

Eigenvectors: A.

$\boxed{\lambda_1 = 1}$

$$(A-I)\vec{v} = \vec{0} \Rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} 2v_1 + v_3 = 0 &\Rightarrow v_1 = -\frac{v_3}{2} \\ 2v_2 + 2v_3 = 0 &\Rightarrow v_2 = -v_3 \\ &\quad v_3 : \text{free} \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \quad \vec{V} = \begin{bmatrix} -V_3/2 \\ -V_3 \\ V_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} V_3$$

Choose: $\quad V_3 = 2 \quad \Rightarrow \quad \boxed{\lambda_1 = 1 \;,\; \vec{V}_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}}$

$$\boxed{W_{\lambda_1 = 1} = \text{Span}\left\{ \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \right\}}$$ Line.

$\boxed{\lambda_2 = 3}$

$$(A - 3I)\vec{V} = \vec{0} \Rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} \boxed{V_3 = 0} \\ 2V_3 = 0 \\ -2V_3 = 0 \end{array} \right\} \Rightarrow$$

V_1, V_2 : free.

$$\Rightarrow \quad \vec{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ 0 \end{bmatrix} = \begin{bmatrix} V_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ V_2 \\ 0 \end{bmatrix} = V_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + V_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boxed{\lambda_2 = 3 \;,\; \vec{V}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} \quad \text{and} \quad \boxed{\lambda_3 = 3 \;,\; \vec{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}$$

$$W_{\lambda_2=3} = \text{Span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} \quad : \text{Plane.}$$

$$\boxed{\left\{ \vec{v}_1 = \begin{bmatrix}-1\\-2\\2\end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix} \right\} \quad l.i.} \Rightarrow$$

$\Rightarrow$ $\boxed{A \text{ is diagonalizable.}}$

<u>Eigenvectors</u> <u>B</u> :

Check: * $\lambda_1 = 1$, $\vec{v}_1 = \begin{bmatrix}0\\-1\\1\end{bmatrix}$; $\boxed{W_{\lambda_1} = \text{Span}\left\{\begin{bmatrix}0\\-1\\1\end{bmatrix}\right\}}$

line.

* $\lambda_2 = \lambda_3 = 3$, $\vec{v}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$; $\boxed{W_{\lambda_2} = \text{Span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}\right\}}$

line.

No other eigenvectors.

B has only two l.i. eigenvectors, not three.

B is <u>not</u> diagonalizable.

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